Norges teknisk-naturvitenskapelige universitet
NTNU
Institutt for fysikk

EKSAMEN I: MNFFY 245 – INNFØRING I KVANTEMEKANIKK

DATO: Fredag 14. desember 2001 TID: 9.00 – 15.00

Antall vekttall: 4
Antall sider: 5

Tillatte hjelpemidler: Kalkulator, matematisk formelsamling.

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Sensurdato: 14. januar 2002.
English version on pages 3 – 5.

Oppgave 1

a. Skriv ned Hamilton-operatoren (H_z) for en partikkel med masse m som beveger seg i det éndimensjonale potensialet $V_z(z) = \frac{1}{2}m\omega^2 z^2$. Anta at systemet ved $t = 0$ prepareres i begynnelsestilstanden

$$\Psi_z(z,0) = C_0 e^{-m\omega z^2/2\hbar}.$$

Bestem C_0 slik at $\Psi_z(z,0)$ er normert, og finn (helst uten regning) forventningsverdien $\langle\, z\,\rangle_0$ ved $t = 0$.

b. Vis at $\Psi_z(z,0)$ er en egenfunksjon til H_z og finn egenverdien. Hva blir da bølgefunksjonen $\Psi_z(z,t)$ for $t > 0$?

c. Betrakt et nytt system, hvor partikkelen beveger seg i det éndimensjonale potensialet $V_x(x) = \frac{1}{2}m\omega^2 x^2$, og anta at begynnelsestilstanden er

$$\Psi_x(x,0) = C_0 e^{-m\omega(x-x_0)^2/2\hbar}.$$

Forklar hvorfor vi kan bruke samme normeringskonstant som ovenfor, og argumentér for at $\langle\, x\,\rangle_0 = x_0$.
Vis (så enkelt som du kan) at forventningsverdien for impulsen p_x ved $t = 0$ er $\langle\, p_x\,\rangle_0 = 0$.
Argumentér for at tilstanden $\Psi_x(x,t)$ er en ikke-stasjonær tilstand (når $x_0 \neq 0$).

d. Vis at usikkerhetene $(\Delta x)_0$ og $(\Delta p_x)_0$ er uavhengige av x_0, og vis at $(\Delta x)_0(\Delta p_x)_0 = \frac{1}{2}\hbar$.
Hva er da $(\Delta z)_0$ og $(\Delta p_z)_0$ for tilstanden $\Psi_z(z,0)$ i **a**?

e. Betrakt enda et nytt system, hvor partikkelen beveger seg i det éndimensjonale potensialet $V_y(y) = \frac{1}{2}m\omega^2 y^2$, og anta at begynnelsestilstanden er

$$\Psi_y(y,0) = C_0 e^{-m\omega y^2/2\hbar} e^{iym\omega x_0/\hbar}.$$

Forklar hvorfor vi kan bruke den samme konstanten C_0 her også.
Argumentér for at $\langle\, y \,\rangle_0$ er lik null og for at $(\Delta y)_0$ har samme verdi som $(\Delta x)_0$ ovenfor.
Finn $\langle\, p_y \,\rangle_0$.

f. Anta nå at partikkelen beveger seg i det tredimensjonale potensialet

$$V(x, y, z) = V_x(x) + V_y(y) + V_z(z) = \tfrac{1}{2} m\omega^2 r^2,$$

og anta at dette systemet prepareres i begynnelsestilstanden

$$\Psi(x, y, z, 0) = \Psi_x(x, 0)\Psi_y(y, 0)\Psi_z(z, 0),$$

hvor $\Psi_x(x, 0)$ osv er begynnelsestilstandene som ble introdusert ovenfor.
Det er mulig å finne bølgefunksjonen $\Psi(x, y, z, t)$ for $t > 0$ for dette systemet, og en kan også vise at usikkerhetene i posisjoner og impulser er tidsuavhengige, men dette skal du ikke gjøre. Bruk i stedet Ehrenfests teorem til å finne forventningsverdiene $\langle\, x \,\rangle_t$, $\langle\, y \,\rangle_t$ og $\langle\, z \,\rangle_t$ for $t > 0$, og vis at $\langle\, \mathbf{r} \,\rangle_t = \hat{\mathbf{e}}_x \langle\, x \,\rangle_t + \hat{\mathbf{e}}_y \langle\, y \,\rangle_t + \hat{\mathbf{e}}_z \langle\, z \,\rangle_t$ følger en sirkelbane.

Oppgitt:

$$I_n(\beta) \equiv \int_{-\infty}^{\infty} x^{2n} e^{-\beta x^2} dx = (-\partial/\partial\beta)^n I_0(\beta)\ ; \qquad I_0(\beta) = (\pi/\beta)^{1/2};$$

$$\frac{d}{dt}\langle\, \mathbf{r} \,\rangle = \frac{\langle\, \mathbf{p} \,\rangle}{m}\ ; \qquad \frac{d}{dt}\langle\, \mathbf{p} \,\rangle = \langle\, -\boldsymbol{\nabla} V \,\rangle.$$

Oppgave 2

a. La F betegne en observabel, $\{f_n\}$ dens egenverdispektrum (som antas å være diskret og ikke-degenerert), og $|n\rangle$ dens normerte egenvektorer.
Betrakt et fysisk system som er i en vilkårlig tilstand $|\psi\rangle$. Ved å bruke fullstendighets-relasjonen kan vi utvikle tilstandsvektoren $|\psi\rangle$ i egenvektorsettet $|n\rangle$ til F:

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle = \sum_n \langle n|\psi\rangle\, |n\rangle.$$

Anta nå at det foretas en måling av observabelen F på dette systemet. (i) Hva er de mulige målte verdiene for F? (ii) Hva er *sannsynlighetsamplituden*, og hva er *sannsynligheten*, for å måle verdien f_n? (iii) Dersom måleresultatet er f_7, hva er da resultatet ved en ny måling av F umiddelbart etter den første målingen?

I resten av denne oppgaven betrakter vi et spinn-$\frac{1}{2}$-system, hvor vi bruker egentilstandene $|\hat{\mathbf{z}}\rangle$ og $|-\hat{\mathbf{z}}\rangle$ som basis. Dette leder til en matriserepresentasjon, hvor basis-tilstandene representeres av Pauli-spinorene

$$\chi_{\hat{\mathbf{z}}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{og} \qquad \chi_{-\hat{\mathbf{z}}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b. Vis eksplisitt at $\chi_{\hat{\mathbf{z}}}$ og $\chi_{-\hat{\mathbf{z}}}$ er egentilstander til $S_z = \frac{1}{2}\hbar\sigma_z$, og bestem egenverdiene.
Finn lengde og retning av vektoren $\langle\, \mathbf{S} \,\rangle = \langle\, S_x \,\rangle \hat{\mathbf{e}}_x + \langle\, S_y \,\rangle \hat{\mathbf{e}}_y + \langle\, S_z \,\rangle \hat{\mathbf{e}}_z$ for tilstanden $\chi_{\hat{\mathbf{z}}}$.

c. Hva er (generelt) de mulige resultatene ved en måling av S_x på spinn-$\frac{1}{2}$-systemet? Hva er sannsynlighetene for å finne hvert av disse resultatene dersom systemet er i tilstanden $\chi_{\hat{\mathbf{z}}}$ før målingen? [Hint: Bruk resultatet for $\langle S_x \rangle$ fra **b**.]
Hva kan du si om tilstanden til systemet umiddelbart *etter* målingen av S_x?

d. Finn en normert løsning $\chi_{\hat{\mathbf{x}}}$ av egenverdiligningen $S_x \chi_{\hat{\mathbf{x}}} = \frac{1}{2}\hbar \chi_{\hat{\mathbf{x}}}$. Hva er den fysiske tolkningen av skalar-produktet $\chi_{\hat{\mathbf{x}}}^\dagger \chi_{\hat{\mathbf{z}}}$? Bruk dette til å verifisere resultatet for en av sannsynlighetene funnet i **b**.

e. Hva er den generelle definisjonen av *spinn-retningen* for spinn-$\frac{1}{2}$-systemet? [Hint: Sjekk hvordan definisjonen virker for tilfellet under **b**.]
Finn spinn-retningen for tilstanden

$$\chi = \begin{pmatrix} 1/\sqrt{2} \\ (1+i)/2 \end{pmatrix}.$$

Oppgitt:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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English version:

Problem 1

a. Write down the Hamiltonian operator (H_z) for a particle with mass m which moves in the one-dimensional potential $V_z(z) = \frac{1}{2}m\omega^2 z^2$. Assume that the system is at $t = 0$ prepared in the initial state

$$\Psi_z(z,0) = C_0 e^{-m\omega z^2/2\hbar}.$$

Determine C_0 such that $\Psi_z(z,0)$ is normalized, and find (preferably without calculation) the expectation value $\langle z \rangle_0$ at $t = 0$.

b. Show that $\Psi_z(z,0)$ is an eigenfunction of H_z and find the eigenvalue. What is then the wavefunction $\Psi_z(z,t)$ for $t > 0$?

c. Consider a new system, where the particle moves in the one-dimensional potential $V_x(x) = \frac{1}{2}m\omega^2 x^2$, and assume that the initial state is

$$\Psi_x(x,0) = C_0 e^{-m\omega(x-x_0)^2/2\hbar}.$$

Explain why we can use the same normalization constant as above, and argue that $\langle\, x\,\rangle_0 = x_0$. Show (in the simplest way you can find) that the expectation value of the momentum p_x at $t = 0$ is $\langle\, p_x\,\rangle_0 = 0$.
Argue that the state $\Psi_x(x,t)$ is a non-stationary state (when $x_0 \neq 0$).

d. Show that the uncertainties $(\Delta x)_0$ and $(\Delta p_x)_0$ are independent of x_0, and show that $(\Delta x)_0(\Delta p_x)_0 = \frac{1}{2}\hbar$. What are then $(\Delta z)_0$ and $(\Delta p_z)_0$ for the state $\Psi_z(z,0)$ in **a**?

e. Consider yet another system, where the particle moves in the one-dimensional potential $V_y(y) = \frac{1}{2}m\omega^2 y^2$, and assume that the initial state is

$$\Psi_y(y,0) = C_0 e^{-m\omega y^2/2\hbar} e^{iym\omega x_0/\hbar}.$$

Explain why we can use the same constant C_0 here too.
Argue that $\langle\, y\,\rangle_0$ is equal to zero and that $(\Delta y)_0$ has the same value as $(\Delta x)_0$ above.
Find $\langle\, p_y\,\rangle_0$.

f. Assume now that the particle moves in the three-dimensional potential

$$V(x,y,z) = V_x(x) + V_y(y) + V_z(z) = \tfrac{1}{2}m\omega^2 r^2,$$

and assume that this system is prepared in the initial state

$$\Psi(x,y,z,0) = \Psi_x(x,0)\Psi_y(y,0)\Psi_z(z,0),$$

where $\Psi_x(x,0)$ etc are the initial states introduced above.
It is possible to find the wavefunction $\Psi(x,y,z,t)$ for $t > 0$ for this system, and one can also show that the uncertainties in positions and momenta are time independent, but you are not supposed to do that. Use, instead, Ehrenfest's theorem to find the expectation values $\langle\, x\,\rangle_t$, $\langle\, y\,\rangle_t$ and $\langle\, z\,\rangle_t$ for $t > 0$, and show that $\langle\, \mathbf{r}\,\rangle_t = \hat{\mathbf{e}}_x\langle\, x\,\rangle_t + \hat{\mathbf{e}}_y\langle\, y\,\rangle_t + \hat{\mathbf{e}}_z\langle\, z\,\rangle_t$ follows a circular orbit.

Given:

$$I_n(\beta) \equiv \int_{-\infty}^{\infty} x^{2n} e^{-\beta x^2} dx = (-\partial/\partial\beta)^n I_0(\beta)\;; \qquad I_0(\beta) = (\pi/\beta)^{1/2};$$

$$\frac{d}{dt}\langle\, \mathbf{r}\,\rangle = \frac{\langle\, \mathbf{p}\,\rangle}{m}\;; \qquad \frac{d}{dt}\langle\, \mathbf{p}\,\rangle = \langle\, -\boldsymbol{\nabla} V\,\rangle.$$

Problem 2

a. Let F denote an observable, $\{f_n\}$ its eigenvalue spectrum (which is assumed to be discrete and non-degenerate), and $|n\rangle$ its normalized eigenvectors.
Consider a physical system which is in an arbitrary state $|\psi\rangle$. Using the completeness relation, we may expand the state vector $|\psi\rangle$ in terms of the eigenvectors $|n\rangle$ of F:

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle = \sum_n \langle n|\psi\rangle\,|n\rangle.$$

Suppose now that a measurement of the observable F is carried out on this system. (i) What are the possible measured values of F? (ii) What is the *probability amplitude*, and what is the *probability*, of measuring the value f_n? (iii) If the measured value is f_7, what is then the result of a new measurement of F immediately after the first measurement?

In the remaining part of this problem, we consider a spin-$\frac{1}{2}$ system, where we use the eigenstates $|\hat{\mathbf{z}}\rangle$ and $|-\hat{\mathbf{z}}\rangle$ as a basis. This leads to a matrix representation, where the basis states are represented by the Pauli spinors

$$\chi_{\hat{\mathbf{z}}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \chi_{-\hat{\mathbf{z}}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b. Show explicitly that $\chi_{\hat{\mathbf{z}}}$ and $\chi_{-\hat{\mathbf{z}}}$ are eigenstates of $S_z = \frac{1}{2}\hbar\sigma_z$, and determine the eigenvalues.
Find the length and direction of the vector $\langle\, \mathbf{S} \,\rangle = \langle\, S_x \,\rangle \hat{\mathbf{e}}_x + \langle\, S_y \,\rangle \hat{\mathbf{e}}_y + \langle\, S_z \,\rangle \hat{\mathbf{e}}_z$ for the state $\chi_{\hat{\mathbf{z}}}$.

c. What are (in general) the possible results of a measurement of S_x on the spin-$\frac{1}{2}$ system? What are the probabilities of getting each of these results if the system is in the state $\chi_{\hat{\mathbf{z}}}$ before the measurement? [Hint: Use the result for $\langle\, S_x \,\rangle$ from **b**.]
What can you say about the state of the system immediately *after* the measurement of S_x?

d. Find a normalized solution $\chi_{\hat{\mathbf{x}}}$ of the eigenvalue equation $S_x\chi_{\hat{\mathbf{x}}} = \frac{1}{2}\hbar\chi_{\hat{\mathbf{x}}}$. What is the physical interpretation of the scalar product $\chi_{\hat{\mathbf{x}}}^\dagger \chi_{\hat{\mathbf{z}}}$? Use this to confirm the result for one of the probabilities found in **b**.

e. What is the general definition of the *spin direction* for the spin-$\frac{1}{2}$ system? [Hint: Check how your definition works for the case in **b**.]
Find the spin direction for the state

$$\chi = \begin{pmatrix} 1/\sqrt{2} \\ (1+i)/2 \end{pmatrix}.$$

Given:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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NTNU – INSTITUTT FOR FYSIKK

EKSAMEN I : MNFFY 45 – Innføring i kvantemekanikk – 14. des. 2001
LØSNINGSFORSLAG

Problem 1

a. The Hamiltonian operator is

$$H_z = \frac{p_z^2}{2m} + V_z(z) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \tfrac{1}{2}m\omega^2 z^2.$$

The normalization integral is

$$1 = \int |\Psi_z(z,0)|^2 dz = |C_0|^2 \int_{-\infty}^{\infty} e^{-m\omega z^2/\hbar} dz = |C_0|^2 \pi^{1/2}(m\omega/\hbar)^{-1/2}.$$

Choosing C_0 real and positive, we find

$$C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.$$

Since the probability density $|\Psi_z(z,0)|^2 = C_0^2 e^{-m\omega z^2/\hbar}$ is symmetric with respect to $z = 0$, the expectation value is

$$\langle\, z \,\rangle_0 = \int z|\Psi_z(z,0)|^2 dz = 0.$$

[Technically, this follows because the integrand is antisymmetric with respect to $z = 0$.]

b. Calculating

$$\frac{\partial \Psi_z(z,0)}{\partial z} = \Psi_z(z,0)(-\frac{m\omega}{\hbar}\, z) \qquad \text{and} \qquad \frac{\partial^2 \Psi_z(z,0)}{\partial z^2} = \Psi_z(z,0)(\frac{m^2\omega^2}{\hbar^2}\, z^2 - \frac{m\omega}{\hbar}),$$

we find that

$$H_z\Psi_z(z,0) = \left[-\frac{\hbar^2}{2m}(\frac{m^2\omega^2}{\hbar^2}\, z^2 - \frac{m\omega}{\hbar}) + \tfrac{1}{2}m\omega^2 z^2\right]\Psi_z(z,0) = \tfrac{1}{2}\hbar\omega\, \Psi_z(z,0).$$

Thus, $\Psi_z(z,0)$ is an eigenfunction of H_z with the eigenvalue $\frac{1}{2}\hbar\omega$, which is the ground-state energy. For $t > 0$, this oscillator will continue to be in the ground state,

$$\Psi(z,t) = \Psi_z(z,0)e^{-iE_0 t/\hbar} = C_0 e^{-m\omega z^2/2\hbar} e^{-i\omega t/2}.$$

c. The probability density $|\Psi_x(x,0)|^2 = C_0^2 e^{-m\omega(x-x_0)^2/\hbar}$ has the same form as in **a**, apart from being shifted such that it is symmetric with respect to $x = x_0$. It follows that we can use the same normalization constant as in **a**, and that

$$\langle\, x \,\rangle_0 = x_0.$$

The expectation value of p_x is

$$\langle\, p_x \,\rangle_0 = \int_{-\infty}^{\infty} \Psi^*(x,0)\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi_x(x,0)dx = -i\hbar \int_{-\infty}^{\infty} \Psi_x(x,0)\frac{\partial}{\partial x}\Psi_x(x,0)dx = 0.$$

[The integral has to be equal to zero because the expectation value is real. Technically, it follows from the facts that $\Psi_x(x,0)$ is symmetric, while $\partial\Psi_x(x,0)/\partial x$ is antisymmetric, with respect to $x = x_0$, such that the integrand is antisymmetric.]

Since the eigenfunctions of H_x are all either symmetric or antisymmetric, whereas the initial state $\Psi_x(x,0)$ is *a*symmetric, it follows that the wavefunction $\Psi(x,t)$ for the present system will be non-stationary (except when $x_0 = 0$).

d. Changing integration variable, from x to $u = x - x_0$, we have

$$\begin{aligned}(\Delta x)_0^2 &= \langle\, (x - \langle\, x\,\rangle_0)^2\,\rangle_0 = \int \Psi_x(x,0)(x-x_0)^2\Psi_x(x,0)dx \\ &= C_0^2\int_{-\infty}^{\infty} u^2 e^{-m\omega u^2/\hbar}du = (\frac{m\omega}{\pi\hbar})^{1/2}\pi^{1/2}\tfrac{1}{2}(\frac{m\omega}{\hbar})^{-3/2} = \frac{\hbar}{2m\omega}.\end{aligned}$$

Thus,

$$(\Delta x)_0 = \sqrt{\frac{\hbar}{2m\omega}}.$$

Furthermore, with

$$\frac{\partial\Psi_x(x,0)}{\partial x} = \Psi_x(x,0)\left[-\frac{m\omega}{\hbar}(x-x_0)\right] \qquad \text{and} \qquad \frac{\partial^2\Psi_x(x,0)}{\partial x^2} = \Psi_x(x,0)\left[\frac{m^2\omega^2}{\hbar^2}(x-x_0)^2 - \frac{m\omega}{\hbar}\right],$$

we find

$$\begin{aligned}(\Delta p_x)_0^2 &= \langle\, (p_x - \langle\, p_x\,\rangle_0)^2\,\rangle_0 = \langle\, p_x^2\,\rangle_0 \\ &= -\hbar^2\int_{-\infty}^{\infty}\Psi^2(x,0)\left[\frac{m^2\omega^2}{\hbar^2}(x-x_0)^2 - \frac{m\omega}{\hbar}\right]dx = -m^2\omega^2(\Delta x)_0^2 + m\omega\hbar = \frac{\hbar m\omega}{2}.\end{aligned}$$

Thus, both $(\Delta x)_0$ and

$$(\Delta p_x)_0 = \sqrt{\frac{\hbar m\omega}{2}}$$

are independent of x_0. For the uncertainty product we find

$$(\Delta x)_0(\Delta p_x)_0 = \tfrac{1}{2}\hbar,$$

which is the minimal value allowed by the uncertainty relation.

Since the z-dependence of $|\Psi_z(z,0)|^2 = C_0^2 e^{-m\omega z^2/\hbar}$ is the same as the x-dependence of $|\Psi_x(x,0)|^2$ for $x_0 = 0$, it follows from the results above that

$$(\Delta z)_0 = (\Delta x)_0 = \sqrt{\frac{\hbar}{2m\omega}} \qquad \text{and} \qquad (\Delta p_z)_0 = (\Delta p_x)_0 = \sqrt{\frac{\hbar m\omega}{2}}.$$

e. We note that the y-dependence of $|\Psi_y(y,0)|^2 = C_0^2 e^{-m\omega y^2/\hbar}$ is the same as the x-dependence of $|\Psi_x(x,0)|^2$ for $x_0 = 0$. Hence, $\Psi_y(y,0)$ is normalized, like $\Psi_x(x,0)$. It also follows that

$$\langle\, y\,\rangle_0 = 0 \qquad \text{and} \qquad (\Delta y)_0 = (\Delta x)_0 = \sqrt{\frac{\hbar}{2m\omega}}.$$

For $\langle\, p_y \,\rangle_0$ we find

$$\begin{aligned}
\langle\, p_y \,\rangle_0 &= \int \Psi^*(y,0)\frac{\hbar}{i}\frac{\partial}{\partial y}\Psi_y(y,0)dy \\
&= \int \Psi^*(y,0)\frac{\hbar}{i}\,\Psi_y(y,0)\left[-\frac{m\omega}{\hbar}\,y + im\omega x_0/\hbar\right]dy \\
&= \int |\Psi_y(y,0)|^2(im\omega y + m\omega x_0)dy = im\omega\langle\, y\,\rangle_0 + m\omega x_0 = m\omega x_0.
\end{aligned}$$

f. From the expression

$$\begin{aligned}
\langle\, x \,\rangle_0 &= \int \Psi^*(x,y,z,0)\, x\, \Psi(x,y,z,0)dxdydz \\
&= \Psi_x^*(x,0)\, x\, \Psi_x(x,0)dx \int |\Psi_y(y,0)|^2 dy \int |\Psi_z(z,0)|^2 dz,
\end{aligned}$$

where the last two integrals are normalization integrals, we understand that the expectation values $\langle\, x\,\rangle_0$ etc are equal to those found above. Thus, the expectation values for $t = 0$ are

$$\begin{aligned}
\langle\, x \,\rangle_0 &= x_0, \qquad & \langle\, p_x \,\rangle_0 = 0, \\
\langle\, y \,\rangle_0 &= 0, \qquad & \langle\, p_y \,\rangle_0 = m\omega x_0, \\
\langle\, z \,\rangle_0 &= 0, \qquad & \langle\, p_z \,\rangle_0 = 0.
\end{aligned}$$

From Ehrenfest's theorem it follows that

$$\frac{d^2\langle\, x\,\rangle_t}{dt^2} = \frac{d}{dt}\frac{\langle\, p_x\,\rangle_t}{m} = \langle -\frac{1}{m}\frac{\partial V}{\partial x}\rangle_t = -\omega^2\langle\, x\,\rangle_t,$$

with the general solution

$$\langle\, x\,\rangle_t = A\cos\omega t + B\sin\omega t, \qquad \langle\, p_x\,\rangle_t = m\frac{d}{dt}\langle\, x\,\rangle_t = -m\omega A\sin\omega t + m\omega B\cos\omega t,$$

and similarly for y and z. Inserting the initial values, we then find that

$$x_0 = A \quad \text{and} \quad 0 = m\omega B \qquad \Longrightarrow \qquad \langle\, x\,\rangle_t = x_0\cos\omega t.$$

Similarly, replacing x by y and at last by z, we find

$$0 = A \quad \text{and} \quad m\omega x_0 = m\omega B \qquad \Longrightarrow \qquad \langle\, y\,\rangle_t = x_0\sin\omega t,$$

and

$$0 = A \quad \text{and} \quad 0 = m\omega B \qquad \Longrightarrow \qquad \langle\, z\,\rangle_t = 0.$$

Thus, $\langle\, \mathbf{r}\,\rangle_t$ follows a circular orbit, with radius x_0.

Problem 2

a. (i) According to one of the basic postulates of quantum mechanics, the possible measured values of the observable F are given by the spectrum of eigenvalues: $\{f_n\}$.
(ii) The *probability amplitude* of measuring the eigenvalue f_n ("or to find the system in the state $|n\rangle$") is given by the "$|n\rangle$"-component of $|\psi\rangle$:

$$A_n = \langle n|\psi\rangle,$$

and the *probability* to get f_n is

$$P_n = |A_n|^2 = |\langle n|\psi\rangle|^2.$$

(iii) According to another postulate, a measurement giving $F = f_7$ will in general change the state of the system, leaving it in the state $|7\rangle$. A new measurement immediately after this measurement will then return the value f_7.

b. We have

$$S_z\chi_{\hat{\mathbf{z}}} = \tfrac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{2}\hbar\chi_{\hat{\mathbf{z}}} \qquad \text{and} \qquad S_z\chi_{-\hat{\mathbf{z}}} = \tfrac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\tfrac{1}{2}\hbar\chi_{-\hat{\mathbf{z}}}.$$

Thus, $\chi_{\hat{\mathbf{z}}}$ and $\chi_{-\hat{\mathbf{z}}}$ are eigenstates of S_z with eigenvalues $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$, respectively. For the state $\chi_{\hat{\mathbf{z}}}$ we find the expectation values

$$\begin{aligned} \langle S_z \rangle &= \chi_{\hat{\mathbf{z}}}^\dagger S_z \chi_{\hat{\mathbf{z}}} = \tfrac{1}{2}\hbar, \\ \langle S_x \rangle &= \chi_{\hat{\mathbf{z}}}^\dagger \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \chi_{\hat{\mathbf{z}}} = 0, \\ \langle S_y \rangle &= \chi_{\hat{\mathbf{z}}}^\dagger \tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \chi_{\hat{\mathbf{z}}} = 0. \end{aligned}$$

Thus, the length of the vector $\langle \mathbf{S} \rangle$ is $\frac{1}{2}\hbar$, and its direction is along the z-axis.

c. The possible results of a measurement of S_x are $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$, as for the measurement of *any* component of $\mathbf{S}$. Since the expectation value of S_x for the state $\chi_{\hat{\mathbf{z}}}$ is zero, the probabilities of getting $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ must be equal, and therefore equal to $\frac{1}{2}$. After the measurement of S_x, the system will be left in the state $\chi_{\hat{\mathbf{x}}}$ if we measure $S_x = \frac{1}{2}\hbar$, and in the state $\chi_{-\hat{\mathbf{x}}}$ if we measure $S_x = -\frac{1}{2}\hbar$.

d. With $S_x = \frac{1}{2}\hbar\sigma_x$, the eigenvalue equation becomes

$$0 = (\sigma_x - \mathbb{1})\chi_{\hat{\mathbf{x}}} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a+b \\ a-b \end{pmatrix}.$$

Thus, the solution is

$$\chi_{\hat{\mathbf{x}}} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Choosing $a = 1/\sqrt{2}$, we get a normalized solution:

$$\chi_{\hat{\mathbf{x}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Physical interpretation: If the system is in the state $\chi_{\hat{\mathbf{z}}}$ and we measure S_x, the the probability amplitude to get $+\frac{1}{2}\hbar$ (and to leave the system in the state $\chi_{\hat{\mathbf{x}}}$) is $\chi_{\hat{\mathbf{x}}}^\dagger\chi_{\hat{\mathbf{z}}}$, and the probability is $|\chi_{\hat{\mathbf{x}}}^\dagger\chi_{\hat{\mathbf{z}}}|^2$. Explicitly, we find for this probability:

$$|\chi_{\hat{\mathbf{x}}}^\dagger\chi_{\hat{\mathbf{z}}}|^2 = \left|(1/\sqrt{2})(1,1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2 = \tfrac{1}{2},$$

confirming the result in **b**.

e. The general definition of the spin direction for the spin-$\frac{1}{2}$ system is that it is given by the vector

$$\langle\, \boldsymbol{\sigma}\, \rangle = \frac{1}{\frac{1}{2}\hbar}\langle\, \mathbf{S}\, \rangle.$$

We note that $\langle\, \boldsymbol{\sigma}\, \rangle = \hat{\mathbf{e}}_z$ for the state $\chi_{\hat{\mathbf{z}}}$ in a, as it should. For the state χ, we find

$$\begin{aligned}
\langle\, \sigma_x\, \rangle &= \chi^\dagger \sigma_x \chi = \left(1/\sqrt{2}, (1-i)/2\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ (1+i)/2 \end{pmatrix} = 1/\sqrt{2}, \\
\langle\, \sigma_y\, \rangle &= \chi^\dagger \sigma_y \chi = \ldots = 1/\sqrt{2}, \\
\langle\, \sigma_z\, \rangle &= \chi^\dagger \sigma_z \chi = \ldots = 0.
\end{aligned}$$

Thus, the spin direction is

$$\langle\, \boldsymbol{\sigma}\, \rangle = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y).$$

We note that it lies in the xy-plane, at an angle of 45 degrees with the x-axis.