(MNFFY-350) (Stjernefysikk)

Eksamen MNFFY 352. Astrofysikk II. 9/12-99

Antydet løsning. Oppgave 1

a) Trykket i en degenerert ideell elektron-gass er (fra kinetisk gass-teori) generelt gitt ved

$$P = (1/3V)\int_0^\infty p v_p f(\varepsilon_p) g(p)\, dp = (N/3V)\langle pv \rangle$$
$$= (1/3)(2/h^3)\int_0^{p_F} [p^2c^2/(p^2c^2+m^2c^4)^{1/2}]\, 4\pi p^2 dp,$$

siden v er (relativistisk) gitt ved

$$v = pc/(p^2+m_e^2c^4)^{1/2} = pc^2/\varepsilon_p,$$
$$\varepsilon_p = (p^2c^2+m_e^2c^4)^{1/2}, \quad \left(\text{og } g(p) = \frac{g_s V 4\pi p^2}{h^3},\ g_s = 2\right)$$

Vi innfører en dimensjonsløs variabel

$$y = p/mc,$$
$$p = mcy,$$

og får trykket lik

$$P = (8\pi m^4c^5/3h^3)\int_0^x y^4 dy/(1+y^2)^{1/2} = (mc^2/\lambda^3)\Phi(x),$$

der

$$\lambda = \hbar/mc, \quad (mc^2/\lambda^3 = m^4c^5/\hbar^3, \quad h^3 = 8\pi^3\hbar^3)$$
$$\Phi(x) = (8\pi^2)^{-1}\{x(1+x^2)^{1/2}[(2x^2/3)-1] + \ln[x+(1+x^2)^{1/2}]\},$$

Partikkel-tettheten n er gitt ved et integral over Fermi-sjøen, dvs. (når $x_F = p_F/mc$):

$$n = (8\pi/3h^3)p_F^3 = (8\pi/3h^3)(m^3c^3x_F^3) = (8\pi m^3c^3/3h^3)x_F^3,$$
$$K = (hc/4)(3/8\pi)^{1/3},$$
$$Kn^{4/3} = (hc/4)(3/8\pi)^{1/3}(8\pi/3)^{4/3}(m^4c^4/h^4)x_F^4 =$$
$$= (\pi m^4c^5/h^3)(2x_F^4/3).$$

Siden

$$m_e c^2/\lambda^3 = m^4 c^5/\hbar^3 = 8\pi^3 m^4 c^5/h^3,$$

ser vi at

$$\underline{\underline{P = (mc^2/\lambda^3)\,\Phi(x) = K \cdot n^{4/3} \cdot I(x)}}$$

b] Rekkeutvikling for $x \ll 1$ gir:

$$x + (1+x^2)^{1/2} = 1 + y$$

$$y = (1+x^2)^{1/2} + x - 1 = x + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \tfrac{1}{16}x^6 \ldots$$

$$\ln[x + (1+x^2)^{1/2}] = \ln(1+y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \ldots$$

$$= x - \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 - \tfrac{5}{112}x^7 + \ldots,$$

dvs. funksjonen blir

$$\Phi(x) = \{x(1+x^2)^{1/2}(\tfrac{2}{3}x^2 - 1) + \ln[x + (1+x^2)^{1/2}]\}/8\pi^2$$

$$= [(\tfrac{2}{3}x^2 - x)(1 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \ldots) + (x - \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 - \tfrac{5}{112}x^7 + \ldots)]/8\pi^2$$

$$= (\tfrac{8}{15}x^5 - \tfrac{4}{21}x^7 + \tfrac{1}{9}x^9 + \ldots)/8\pi^2 = \underline{\underline{(x^5 - \tfrac{5}{14}x^7 + \tfrac{5}{24}x^9 + \ldots)/15\pi^2}}$$

Rekkeutvikling for $x \gg 1$ gir tilsvarende

$$(1+x^2)^{1/2} = x[1 + (1/x)^2]^{1/2} = x[1 + \tfrac{1}{2}(1/x)^2 - \tfrac{1}{8}(1/x)^4 + \ldots],$$

$$x + (1+x^2)^{1/2} = x\{1 + [1 + (1/x)^2]^{1/2}\} = x[2 + \tfrac{1}{2}(1/x)^2 - \tfrac{1}{8}(1/x)^4 + \ldots]$$

$$= 2x[1 + \tfrac{1}{4}(1/x)^2 - \tfrac{1}{16}(1/x)^4 + \ldots]$$

$$\ln[x + (1+x^2)^{1/2}] = \ln(2x) + \ln(1+y) = \ln(2x) + y - \tfrac{1}{2}y^2 + \ldots,$$

$$y = \tfrac{1}{4}(1/x)^2 - \tfrac{1}{16}(1/x)^4 + \ldots,$$

dvs. funksjonen blir

$$\Phi(x) = \{x(1+x^2)^{1/2}(\tfrac{2}{3}x^2 - 1) + \ln[x + (1+x^2)^{1/2}]\}/8\pi^2$$

$$= \{(\tfrac{2}{3}x^2 - 1)x^2[1 + \tfrac{1}{2}(1/x)^2 + \tfrac{1}{8}(1/x)^4 + \ldots] + \ln(2x) + \ldots\}/8\pi^2$$

$$= [\tfrac{2}{3}x^4 - \tfrac{1}{3}x^2 + \ln(2x) + \ldots]/8\pi^2 = \underline{[x^4 - x^2 + \tfrac{3}{2}\ln(2x) \ldots]/12\pi^2}$$

c] Fra rekkeutviklingen under b] ser vi at ikke-relativistisk får vi

$$\Phi(x_F) \to x_F^5/15\pi^2,$$
$$P_{NR} = (mc^2/\lambda^3)(x_F^5/15\pi^2)$$
$$= (8\pi^3 m^4 c^5/h^3)\, n^{5/3} (3/8\pi)^{5/3} (h^5/m^5c^5)/15\pi^2$$
$$\underline{\underline{= (h^2/5m)(3/8\pi)^{2/3} \cdot n^{5/3} = K_{NR} \cdot n^{5/3}}},$$

og ultra-relativistisk får vi

$$\Phi(x_F) \to x^4/12\pi^2,$$
$$P_{UR} = (mc^2/\lambda^3)(x^4/12\pi^2)$$
$$= (8\pi^3 m^4 c^5/h^3)\, n^{4/3} (3/8\pi)^{4/3} (h^4/m^4c^4)/12\pi^2$$
$$\underline{\underline{= (hc/4)(3/8\pi)^{1/3} \cdot n^{4/3} = K_{UR} \cdot n^{4/3}}}$$

Oppgave 2

a] Reaksjonshastigheten R_{AB} er gitt ved

$$R_{AB} \sim S(E_0) \exp[-3(E_G/4kT)^{1/3}],$$

der

$$E_G = (\pi\alpha Z_A Z_B)^2 \cdot 2m_r c^2,$$
$$m_r = m_A m_B/(m_A + m_B).$$

For CNO-syklusen må vi betrakte reaksjonene

$$p + {}^{12}C \to {}^{13}N + \gamma, \quad S(0) = 1.5 \text{ keV} \cdot \text{barn},$$
$$p + {}^{13}C \to {}^{14}N + \gamma, \quad S(0) = 5.5 \text{ keV} \cdot \text{barn},$$
$$p + {}^{14}N \to {}^{15}O + \gamma, \quad S(0) = 3.3 \text{ keV} \cdot \text{barn},$$
$$p + {}^{15}N \to {}^{12}C + {}^{4}He, \quad S(0) = 78 \text{ keV} \cdot \text{barn}.$$

Tilsvarende reduserte masser blir

$m_r(p+{}^{12}C) = m_p \cdot 12/(1+12) = (12/13)\,m_p$,
$m_r(p+{}^{13}C) = m_p \cdot 1 \cdot 13/(1+13) = (13/14)\,m_p$,
$m_r(p+{}^{14}N) = m_p \cdot 1 \cdot 14/(1+14) = (14/15)\,m_p$,
$m_r(p+{}^{15}N) = m_p \cdot 1 \cdot 15/(1+15) = (15/16)\,m_p$,

og Gamow-energien blir

$$E_G(p+{}^{12}C) = (\pi \cdot 6/137)^2\, 2\,(12/13) \cdot 1.67 \cdot 10^{-27} (3 \cdot 10^8)^2\ \mathrm{J}$$
$$= \underline{5.253 \cdot 10^{-12}\ \mathrm{J}} = 5.253 \cdot 10^{-12}/(1.6 \cdot 10^{-19})\ \mathrm{eV}$$
$$= 32830\ \mathrm{keV} = \underline{32.83\ \mathrm{MeV}},$$
$$E_G(p+{}^{13}C) = (\pi \cdot 6/137)^2\, 2\,(13/14) \cdot 1.67 \cdot 10^{-27} (3 \cdot 10^8)^2\ \mathrm{J}$$
$$= \underline{5.284 \cdot 10^{-12}\ \mathrm{J}} = 5.284 \cdot 10^{-12}/(1.6 \cdot 10^{-19})\ \mathrm{eV}$$
$$= 33030\ \mathrm{keV} = \underline{33.03\ \mathrm{MeV}},$$
$$E_G(p+{}^{14}N) = (\pi \cdot 7/137)^2\, 2\,(14/15) \cdot 1.67 \cdot 10^{-27} (3 \cdot 10^8)^2\ \mathrm{J}$$
$$= \underline{7.229 \cdot 10^{-12}\ \mathrm{J}} = 7.229 \cdot 10^{-12}/(1.6 \cdot 10^{-19})\ \mathrm{eV}$$
$$= 45180\ \mathrm{keV} = \underline{45.18\ \mathrm{MeV}}$$
$$\underline{\underline{E_G(p+{}^{15}N)}} = (\pi \cdot 7/137)^2\, 2\,(15/16) \cdot 1.67 \cdot 10^{-27} (3 \cdot 10^8)^2\ \mathrm{J}$$
$$= \underline{7.261 \cdot 10^{-12}\ \mathrm{J}} = 7.261 \cdot 10^{-12}/(1.6 \cdot 10^{-19})\ \mathrm{eV}$$
$$= 45380\ \mathrm{keV} = \underline{\underline{45.38\ \mathrm{MeV}}}\ \text{(som er størst)}$$

b) For

$T = 1.5 \cdot 10^7\ \mathrm{K}$,
$kT = 8.62 \cdot 10^{-5} \cdot 1.5 \cdot 10^7\ \mathrm{eV} = 1290\ \mathrm{eV} = 1.3\ \mathrm{keV}$,

vil reaksjonshastighetene være proporsjonale med

$$R_{AB}(p+{}^{12}C) \sim 1.5 \exp[-3(32830/5.2)^{1/3}] \sim 1.5 \exp(-55.45)$$
$$\sim 1.5 \cdot 8.29 \cdot 10^{-25} \sim \underline{1.24 \cdot 10^{-24}}$$
$$R_{AB}(p+{}^{13}C) \sim 5.5 \exp[-3(33030/5.2)^{1/3}] \sim 5.5 \exp(-55.56)$$
$$\sim 5.5 \cdot 7.42 \cdot 10^{-25} \sim \underline{4.08 \cdot 10^{-24}}$$

$$R_{AB}(p+{}^{14}N) \sim 3.3\exp[-3(45180/5.2)^{1/3}] \sim 3.3\exp(-61.67)$$
$$\sim 3.3\cdot 1.65\cdot 10^{-27} \sim \underline{5.45\cdot 10^{-27}}$$
$$R_{AB}(p+{}^{15}N) \sim 78\cdot\exp[-3(45380/5.2)^{1/3}] \sim 78\cdot\exp(-61.76)$$
$$\sim 78\cdot 1.51\cdot 10^{-27} \sim \underline{1.18\cdot 10^{-25}}$$

Den mest langsomme reaksjonen med minst R_{AB} er da

$$\underline{\underline{p + {}^{14}N \rightarrow {}^{15}O + \gamma}}$$

c) Temperaturavhengigheten er gitt ved

$$R_{AB}(T) \sim (E_G/4kT)^{2/3}\exp[-3(E_G/4kT)^{1/3}]$$

som gir oss

$$dR_{AB}/dT \approx \{(E_G/4k)^{2/3}(-\tfrac{2}{3}T^{-5/3}) + (E_G/4kT)^{2/3}[-3(\tfrac{E_G}{4k})^{1/3}(-\tfrac{1}{3}T^{-4/3})]\}$$
$$\times\exp[-3(E_G/4kT)^{1/3}]$$
$$= [-\tfrac{2}{3}(E_G/4k)^{2/3}T^{-5/3} + (E_G/4k)T^{-2}]\exp[-3(E_G/4kT)^{1/3}]$$
$$= R_{AB}[-(2/3) + (E_G/4kT)^{1/3}]T^{-1}\exp[-3(E_G/4kT)^{1/3}],$$

$$dR_{AB}/R_{AB} \approx [(E_G/4kT)^{1/3} - (2/3)]\,dT/T,$$
$$\ln(R_{AB}) \approx [(E_G/4kT)^{1/3} - (2/3)]\ln T,$$

der

$$(E_G/4kT)^{1/3} = (45180/5.2)^{1/3} = 8688^{1/3} = 20.6,$$

dvs.

$$\ln(R_{AB}) \approx 20\cdot\ln(T), \qquad \underline{\underline{R_{AB} = R_{CNO} \sim T^{20}}}$$

For reaksjonen

$$p + d \rightarrow {}^3He + \gamma,$$

får vi en redusert masse lik

$$m_r(p+d) = m_p\cdot 1\cdot 2/(1+2) = (2/3)m_p,$$

og Gamow-energien blir

$$E_G(p+d) = (\pi/137)^2 \cdot 2(2/3)1.67 \cdot 10^{-27}(3 \cdot 10^8)^2\,J$$
$$= 1.05 \cdot 10^{-13}\,J = 1.05 \cdot 10^{-13}/(1.6 \cdot 10^{-19})\,eV$$
$$= \underline{\underline{656\ keV}},$$

dvs.

$$(E_G/4kT)^{1/3} = (656/5.2)^{1/3} = 126^{1/3} \approx 5,$$

dvs.

$$\ln R_{AB} \sim [(E_G/4kT)^{1/3} - (2/3)] \ln T \sim 4.3 \ln T$$
$$\underline{\underline{R_{AB} = R_{pp} \sim T^{4.3}}}$$

Oppgave 3

a) Relativistisk enpartikkel-energi er

$$\varepsilon = \sqrt{(pc)^2 + (mc^2)^2},$$

og total energi blir

$$E = 2\sum_{p<p_F} \sqrt{(pc)^2 + (mc^2)^2} = (2\Omega/h^3)\int_0^{p_F} 4\pi p^2 \sqrt{(pc)^2 + (mc^2)^2}\,dp,$$

der Fermi-impulsen er gitt ved (når Ω er volum)

$$n = N/\Omega = k_F^3/3\pi^2,$$
$$p_F = \hbar(3\pi^2 N/\Omega)^{1/3},$$

Vi innfører en ny variabel

$$x = p/mc,$$

dvs.

$$E = \Omega(m^4c^5/\pi^2\hbar^3) f(x_F) = N(m^4c^5/\pi^2\hbar^3) f(x_F)/n,$$

der
$$f(x_F) = \int_0^{x_F} x^2\sqrt{1+x^2}\,dx$$
$$x_F = p_F/mc = (\hbar/mc)(3\pi^2 n)^{1/3}.$$

For $x_F \gg 1$ er $f(x_F) \approx \int_0^{x_F} x^3[1+(2x^2)^{-1}]dx = \int_0^{x_F}(x^3+\frac{1}{2}x)dx$,
dvs. $f(x_F) = \frac{1}{4}x_F^4(1+x_F^{-2}+\dots) = \frac{1}{4}(x_F^4+x_F^2+\dots)$.

Stjernas masse og radius er
$$M = Nm,$$
$$R = (3\Omega/4\pi)^{1/3},$$
dvs.
$$n = N/\Omega = 3M/(4\pi m R^3) = (m^3c^3/3\pi^2\hbar^3)(\bar{M}/\bar{R}^3),$$
$$x_F = (\hbar/mcR)(9\pi M/4m)^{1/3} = \bar{M}^{1/3}/\bar{R},$$
$$\bar{M} = 9\pi M/4m,$$
$$\bar{R} = mcR/\hbar.$$

Total energi blir da
$$E = \Omega(m^4c^5/\pi^2\hbar^3)f(x_F) = (mc^2/3\pi)(\bar{M}^{4/3}/\bar{R} + \bar{M}^{2/3}\bar{R}),$$
dvs. trykket er
$$P = -\partial E/\partial\Omega = -(\partial\bar{R}/\partial\Omega)(\partial E/\partial\bar{R}) = (m^4c^5/12\pi^2\hbar^3)(\bar{M}^{4/3}/\bar{R}^4 - \bar{M}^{2/3}/\bar{R}^2)$$

b) Bindingsenergien er bundet potensiell gravitasjons-energi som tilsvarer arbeidet med å føre masse dm inntil en kule med masse m(r), og vi får energiforandring dU og masseøkning dm:
$$dU = -Gm(r)\,dm/r,$$
$$dm = 4\pi\rho r^2 dr,$$
$$m(r) = (4\pi r^3/3)\rho,$$
dvs.

$$dU = -\tfrac{1}{3}(4\pi)^2 \rho^2 G r^4 dr.$$
$$M = (4\pi/3) R^3 \rho.$$

~~Total energi er da~~

$$U(R) = -\int_0^R \tfrac{1}{3}(4\pi)^2 \rho^2 G r^4 dr = -(16\pi^2/15)\rho^2 G R^5 = -3GM^2/5R,$$

og trykket blir

$$P = -\partial U/\partial \Omega = -(\partial R/\partial \Omega)(\partial U/\partial R) = \underline{3GM^2/20\pi R^4}$$
$$= \underline{\underline{(4Gm^6c^4/135\pi^3\hbar^4)\bar{M}^2/\bar{R}^4}}.$$

c) Likevektsbetingelsen blir nå

$$C(\bar{M}^{4/3}/\bar{R}^4 - \bar{M}^{2/3}/\bar{R}^2) = C'\bar{M}^2/\bar{R}^4,$$

der

$$C = m^4c^5/12\pi^2\hbar^3,$$
$$C' = (3G/20\pi)(4m/9\pi)^2(mc/\hbar)^4 = 4Gm^6c^4/135\pi^3\hbar^4,$$

dvs.

$$\underline{\underline{\bar{R} = \bar{M}^{1/3}\sqrt{1-(\bar{M}/\bar{M}_0)^{2/3}}}},$$

der

$$\bar{M}_0 = (C/C')^{3/2} = (45\pi\hbar c/16Gm^2)^{3/2}.$$

Vi ser at $\bar{R} \to 0$ når $\bar{M} \to \bar{M}_0$, dvs.

$$R \to 0 \text{ når } M \to M_0 = 4m\bar{M}_0/9\pi = \underline{\underline{(4m/9\pi)(45\pi\hbar c/16Gm^2)^{3/2}}}$$

dvs. <u>M_0 blir en øvre grense</u> (siden større masse vil kollapse ...)

$$\left(M_0 = \frac{4\cdot 1.67\cdot 10^{-24}(45\pi\cdot 1.05\cdot 10^{-27}\cdot 3\cdot 10^{10})^{3/2}}{9\pi\,[16\cdot 6.67\cdot 10^{-8}(1.67\cdot 10^{-24})^2]^{3/2}}\, g \approx 6M_\odot\right)$$