Eksamen. MNFFY-~~XX~~3/V99. Hvite dverger. 4/6-99
(MNFFY-450. Kompakte stjerner)

Antydet løsning. Oppgave 1

a] Ved likevekt gjelder

$$\mu_e + \mu_p = \mu_n ,$$

for de kjemiske potensialene, når vi setter

$$\mu_\nu \approx 0 .$$

Med

$$\mu = \varepsilon_F = [(p_F c)^2 + (m c^2)^2]^{1/2} = mc^2[1 + (p_F/mc)^2]^{1/2} ,$$

$$x = p_F/mc ,$$

for henholdsvis elektron, proton og neytron, får vi

$$m_e(1+x_e^2)^{1/2} + m_p(1+x_p^2)^{1/2} = m_n(1+x_n^2)^{1/2} .$$

Ladningsneytralitet krever

$$n_e = x_e^3/3\pi^2\lambda_e^3 = n_p = x_p^3/3\pi^2\lambda_p^3 ,$$

$$m_e x_e = m_p x_p ,$$

der $\hbar/mc = \lambda$

er Compton-bølgelengden. Blandingsforholdet mellom partiklene finnes nå ved innsetting i ligningen for de kjemiske potensialene, dvs.

$$\underline{\underline{(m_e^2 + m_p^2 x_p^2)^{1/2} + m_p(1+x_p^2)^{1/2} = m_n(1+x_n^2)^{1/2} .}}$$

b] Kvadrering av det siste uttrykket under a] gir da

$$m_e^2 + m_p^2 x_p^2 = [m_n(1+x_n^2)^{1/2} - m_p(1+x_p^2)^{1/2}]^2$$

$$= m_n^2 + m_n^2 x_n^2 - 2m_n m_p(1+x_n^2)^{1/2}(1+x_p^2)^{1/2} + m_p^2 x_p^2 + m_p^2 ;$$

$$(m_n^2 + m_p^2 - m_e^2 + m_n^2 x_n^2)^2 = 4m_n^2 m_p^2 (1+x_n^2)(1+x_p^2)$$

$$= 4m_n^2 m_p^2 (1 + x_n^2 + x_p^2 + x_n^2 x_p^2) ,$$

$$(m_n^2 + m_p^2 - m_e^2)^2 + 2m_n^2 x_n^2 (m_n^2 + m_p^2 - m_e^2) + m_n^4 x_n^4$$

$$= 4m_n^2 m_p^2 x_p^2 (1+x_n^2) + 4m_n^2 m_p^2 (1+x_n^2) ,$$

$$4\, m_n^2 m_p^2 x_p^2 (1+x_n^2) = (Q^2 - m_e^2)\left[(m_n+m_p)^2 - m_e^2\right] + 2 m_n^2 x_n^2 (m_n^2 - m_p^2 - m_e^2) + m_n^4 x_n^4 ,$$

siden
$$Q^2 - m_e^2 = (m_n - m_p)^2 - m_e^2 ,$$

dvs.
$$n_p/n_n = (m_p x_p / m_n x_n)^3$$
$$= \left[\left\{1 + 2(m_n^2 - m_p^2 - m_e^2)/m_n^2 x_n^2 + (Q^2 - m_e^2) \times \left[(m_n+m_p)^2 - m_e^2\right]/m_n^4 x_n^4\right\}/(1 + 1/x_n^2)\right]^{3/2}/8$$
$$\underline{\underline{\approx \left\{\left[1 + 4Q/m_n x_n^2 + 4(Q^2 - m_e^2)/m_n^2 x_n^4\right]/(1+1/x_n^2)\right\}^{3/2}/8 ,}}$$

siden
$$Q = m_n - m_p \ll m_p \approx m_n ,$$
$$m_e \ll m_p \approx m_n ,$$
$$m_n^2 - m_p^2 - m_e^2 = (m_n - m_p)(m_n + m_p) - m_e^2 \approx 2Q m_n .$$

c) Vi får minimum for
$$\partial(n_p/n_n)/\partial x_n = 0 , \qquad (n_n = x_n^3/3\pi^2 \lambda_n^3)$$

dvs. for
$$\partial\left\{\left[x_n^4 + 4Q x_n^2/m_n + 4(Q^2 - m_e^2)/m_n^2\right]/(x_n^4 + x_n^2)\right\}/\partial x_n = 0 ,$$
$$(4x_n^3 + 8Q x_n/m_n)(x_n^4 + x_n^2) - \left[x_n^4 + 4Q x_n^2/m_n + 4(Q^2 - m_e^2)/m_n^2\right] \times (4x_n^3 + 2x_n) = 0 ,$$
$$4x_n^6 + 8Q x_n^4/m_n + 4x_n^4 + 8Q x_n^2/m_n - 4x_n^6 - 16 Q x_n^4/m_n - 16(Q^2 - m_e^2) x_n^2/m_n^2 - 2x_n^4 - 8Q x_n^2/m_n - 8(Q^2 - m_e^2)/m_n^2 = 0 ,$$
$$\underline{\underline{n_n \approx (2^{3/2}/3\pi^2 \lambda_n^3)\left[(Q^2 - m_e^2)/m_n^2\right]^{3/4}}} , \text{ dvs. } x_n \approx 2^{1/2}\left[(Q^2 - m_e^2)/m_n^2\right]^{1/4} ,$$

som gir
$$\underline{\underline{(n_p/n_n)_{min} = \left\{(Q/m_n) + \left[(Q^2 - m_e^2)^{1/2}/m_n\right]\right\}^{3/2}}} \quad \begin{pmatrix} Q \ll m_n \\ x_n \ll 1 \end{pmatrix}$$

For meget store tettheter vil da
$$x_n \to \infty , \text{ for } \rho_0 \to \infty ,$$
$$(n_p/n_n) \to 1/8 , \text{ for } x_n \to \infty .$$

Dessuten gjelder
$$n_e = n_p$$
ved ladningsnøytralitet, dvs.
$$\underline{\underline{n_e : n_p : n_n \to 1 : 1 : 8 , \text{ for } \rho_0 \to \infty}}$$

Antydet løsning. Oppgave 2

a) Ligningene

$$dm/dr = 4\pi r^2 \rho,$$
$$dP/dr = -Gm\rho/r^2,$$
$$P = K\rho^{\Gamma};$$

kan kombineres til

$$m = -(r^2/\rho G)(dP/dr),$$
$$dm/dr = -(1/G)\,d[(r^2/\rho)/(dP/dr)]/dr = 4\pi r^2 \rho,$$
$$dP/dr = \Gamma K \rho^{\Gamma-1}(d\rho/dr),$$

dvs.

$$r^{-2} d[(r^2/\rho)(dP/dr)]/dr = (\Gamma K/r^2)\,d[r^2 \rho^{\Gamma-2}(d\rho/dr)]/dr = -4\pi G\rho,$$
$$r^{2} d[r^2 \rho^{\Gamma-2}(d\rho/dr)]/dr = -(4\pi G/\Gamma K)\rho$$
$$= a^{-2}\xi^{-2} d[a^2\xi^2 \rho_c^{\Gamma-2}\theta^{(\Gamma-2)n}(d\rho/a\,d\xi)]/a\,d\xi$$
$$= a^{-3}\xi^{-2} d[\rho_c^{\Gamma-2} a\,\xi^2\theta^{n(\Gamma-2)}(d\rho/d\xi)]/d\xi$$
$$= (\rho_c^{\Gamma-2}/a^2\xi^2)\,d[\xi^2\theta^{n(\Gamma-2)}\rho_c n\theta^{n-1}(d\theta/d\xi)]/d\xi$$
$$= (n\rho_c^{\Gamma-1}/a^2\xi^2)\,d[\xi^2\theta^{n(\Gamma-1)-1}(d\theta/d\xi)]/d\xi = -(4\pi G/\Gamma K)\rho_c\theta^n;$$
$$\xi^{-2} d[\xi^2\theta^{n(\Gamma-1)-1}(d\theta/d\xi)]/d\xi = -(4\pi G a^2/\Gamma K n \rho_c^{\Gamma-2})\theta^n.$$

Tilpasning til Lane-Emdens ligning gir oss da betingelsene

$$n\Gamma - n - 1 = 0,$$
$$4\pi G a^2/\Gamma K n \rho_c^{\Gamma-2} = 1,$$

dvs.

$$\underline{\Gamma = (n+1)/n = 1 + (1/n)}, \qquad \underline{n = 1/(\Gamma - 1)},$$
$$\underline{\underline{a = (\Gamma K n \rho_c^{\Gamma-2}/4\pi G)^{1/2} = [(n+1)K\rho_c^{(1-n)/n}/4\pi G]^{1/2}}},$$
$$\underline{\underline{\xi^{-2} d[\xi^2 (d\theta/d\xi)]/d\xi = -\theta^n}}.$$

b) Total masse er gitt ved

$$M = \int_0^R 4\pi r^2 \rho\,dr = \int_0^{R/a} 4\pi a^3 \xi^2 \rho_c \theta^n d\xi = 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2\theta^n d\xi$$
$$= -4\pi a^3 \rho_c \int_0^{\xi_1} [d(\xi^2 d\theta/d\xi)/d\xi]\,d\xi = 4\pi a^3 \rho_c \xi_1^2 \,|\theta'(\xi_1)|,$$

dvs.
$$\underline{\underline{M = 4\pi\left[(n+1)K/4\pi G\right]^{3/2}\rho_c^{(3-n)/2n}\xi_1^2\,|\theta'(\xi_1)|}}\,.$$

Total radius blir
$$\underline{\underline{R = a\xi_1 = \left[(n+1)K/4\pi G\right]^{1/2}\cdot\rho_c^{(1-n)/2n}\cdot\xi_1}}\,.$$

c) Fra uttrykket for total masse M, finner vi at forholdet mellom midlere tetthet $\bar{\rho}$ og sentral tetthet ρ_c blir lik
$$\bar{\rho}/\rho_c = (3M/4\pi R^3)/\rho_c$$
$$= 3\cdot 4\pi a^3\rho_c\xi_1^2|\theta'(\xi_1)|/(4\pi a^3\xi_1^3\rho_c) = \underline{\underline{3|\theta'(\xi_1)|/\xi_1}},$$
siden
$$M = (4\pi R^3/3)\bar{\rho}.$$
Fra uttrykket for total radius R, finner vi at
$$R \sim \rho_c^{(1-n)/2n}\xi_1,$$
$$\rho_c \sim R^{2n/(1-n)},$$
og fra uttrykket for total M får vi
$$M \sim a^3\rho_c\xi_1^2 \sim \rho_c^{(3-n)/2n} \sim R^{(3-n)/(1-n)}.$$
For
$$\Gamma = 5/3$$
får vi da
$$n = 1/(\Gamma - 1) = 3/2,$$
$$\underline{\underline{M \sim R^{-(3/2)/(1/2)} = R^{-3}}}$$

Antydet løsning, Oppgave 3.

a] Poissons ligning gir

$$\nabla^2\Phi = 4\pi G\rho,$$

dvs. $$\nabla^2\delta\Phi = 4\pi G\delta\rho,$$

der $$\delta\rho = -\nabla\cdot(\rho\xi),$$

dvs. $$\delta\Phi = -G\int\frac{\delta\rho'}{|\underline{x}-\underline{x}'|}d^3x' = G\int\frac{\nabla'\cdot(\rho'\xi')}{|\underline{x}-\underline{x}'|}d^3x' = -G\int\rho'\xi'\nabla'\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)d^3x',$$

ifølge delvis integrasjon, siden

$$\rho' = 0 \text{ for } x' \geq R,$$

på overflaten. Dessuten er

$$\nabla\left(\frac{1}{|\underline{x}-\underline{x}'|}\right) = -\nabla'\left(\frac{1}{|\underline{x}-\underline{x}'|}\right),$$

$$\nabla^2\left(\frac{1}{|\underline{x}-\underline{x}'|}\right) = -4\pi\delta(\underline{x}-\underline{x}').$$

Derivasjon av uttrykket for $\delta\Phi$ gir da

$$\begin{aligned}\nabla\delta\Phi &= \nabla\left(G\int\frac{\nabla'(\rho'\xi')}{|\underline{x}-\underline{x}'|}d^3x'\right) = G\int\nabla'(\rho'\xi')\nabla\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)d^3x' \\ &= -G\int\nabla'(\rho'\xi')\nabla'\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)d^3x' \\ &= -G\Big[\rho'\xi'\nabla'\left(\frac{1}{\underline{x}-\underline{x}'}\right) + G\int\rho'\xi'\nabla'^2\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)d^3x' \\ &= -4\pi G\int\rho'\xi'\delta(x-x')d^3x' = -4\pi G\rho\xi,\end{aligned}$$

dvs. $$\underline{\underline{\nabla_i\delta\Phi = -4\pi G\rho\xi_i,}}$$

siden $$\nabla_i\delta\Phi = (x_i/r)\nabla\delta\Phi,$$
$$\xi_i = (x_i/r)\xi.$$

b] Ligningen for hydrostatisk likevekt følger egentlig fra ligningen for impulsbevarelse, som gir

$$\nabla P + \rho\nabla\Phi = 0,$$

og ligningen for radielle oscillasjoner på grunn av små perturbasjoner av en statisk likevekt ~~for~~ for en kulesymmetrisk stjerne, blir da

$$\sum_j L_{ij}\xi^j = \sum_j \nabla_i(\Gamma_1 P \nabla_j \xi^j) - (\nabla_j \xi^j)\nabla_i P + (\nabla_i \xi^j)\nabla_j P$$
$$- \rho\xi^j \nabla_j \nabla_i \Phi - \rho \nabla_i \delta\Phi$$
$$= d\{\Gamma_1 P[d(r^2\xi)/dr]/r^2\}/dr - [d(r^2\xi)/dr](dP/dr)/r^2$$
$$+ (d\xi/dr)(dP/dr) - \rho\xi d^2\Phi/dr^2 + 4\pi G\rho^2\xi$$
$$= d\{\Gamma_1 P[d(r^2\xi)/dr]/r^2\}/dr - [(2/r)\xi + (d\xi/dr)](dP/dr)$$
$$+ (d\xi/dr)(dP/dr) - \rho\xi[\nabla^2\Phi - (2/r)(d\Phi/dr)] + 4\pi G\rho^2\xi$$
$$= d\{\Gamma_1 P[d(r^2\xi)/dr]/r^2\}/dr - (2/r)(dP/dr)\xi - 4\pi G\rho^2\xi$$
$$- (2/r)(dP/dr)\xi + 4\pi G\rho^2\xi$$
$$= d\{\Gamma_1 P[d(r^2\xi)/dr]/r^2\}/dr - (4/r)(dP/dr)\xi,$$

som gir oss ligningen

$$\underline{d\{\Gamma_1 P[d(r^2\xi)/dr]/r^2\} - (4/r)(dP/dr)\xi + \omega^2\rho\xi = 0.}$$

c) For $\Gamma_1 = 4/3$,

får vi egenverdi-ligningen

$$d\{\Gamma_1 P[d(r^2\xi)/dr]/r^2\} - (4/r)(dP/dr)\xi + \omega^2\rho\xi$$
$$= d\{(4P/3)[d(K\cdot r^3)/dr]/r^2\} - (4/r)(dP/dr)Kr + \omega^2\rho r$$
$$= d(4KP)/dr - 4K(dP/dr) + \omega^2\rho r = \omega^2\rho r = 0,$$

dvs.

$$\underline{\omega^2 = 0},$$

for $\underline{\xi = K\cdot r = \text{konstant}\cdot r}$.