SIF 4003. Eksamen 6/12-1999

Oppgave 1

a) $m\frac{d^2x}{dt^2} = -kx$

$$\underline{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

$$x(t) = A_0 \sin(\omega t + \varphi)$$

$$\frac{dx}{dt} = \omega A_0 \cos(\omega t + \varphi)$$

$$\frac{d^2x}{dt^2} = -\omega^2 A_0 \sin(\omega t + \varphi)$$

Innsatt i ligningen:

$$-\omega^2 A_0 \sin(\omega t + \varphi) + \frac{k}{m} A_0 \sin(\omega t + \varphi) = 0$$

Vi ser at $x(t) = A_0 \sin(\omega t + \varphi)$ er en løsning hvis $\underline{\omega^2 = \frac{k}{m}}$

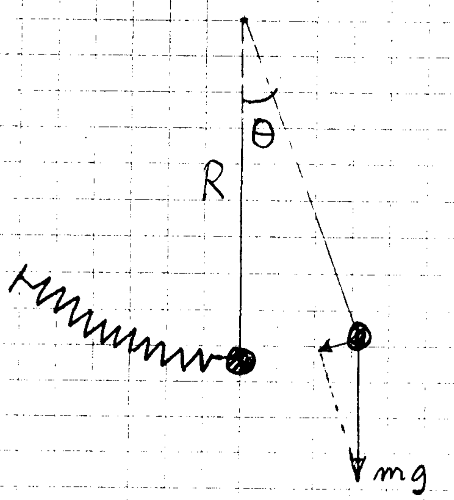
$t = 0$: $X = A_0 = A_0 \sin\varphi$ $\Rightarrow \underline{\varphi = \frac{\pi}{2} = 90^\circ}$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{7}{0{,}020}}\ \text{Hz} = \underline{2{,}98\ \text{Hz}}$$

$$T = \frac{1}{f} = \frac{1}{2{,}98}\ \text{s} = \underline{0{,}34\ \text{s}}$$

$$A_0 = \underline{0{,}10\ \text{m}}$$

b)



$x = R \cdot \theta \qquad g = \text{tyngdeakselerasjonen}$

$$m \cdot \frac{d^2x}{dt^2} = -k \cdot R\theta - mg \sin\theta$$

$$\sin\theta \approx \theta$$

$$\underline{\frac{d^2\theta}{dt^2} + \left(\frac{k}{m} + \frac{g}{R}\right)\theta = 0} \qquad \text{I}$$

$$\theta = B_0 \sin(\omega t + \gamma)$$

$$\frac{d\theta}{dt} = \omega B_0 \cos(\omega t + \gamma)$$

$$\frac{d^2\theta}{dt^2} = -\omega^2 B_0 \sin(\omega t + \gamma) \quad \text{Innsatt i ligningen}$$

$$-\omega^2 B_0 \sin(\omega t + \gamma) + \left(\frac{k}{m} + \frac{g}{R}\right) B_0 \sin(\omega t + \gamma) = 0$$

Vi ser at $\theta = B_0 \sin(\omega t + \gamma)$ er en løsning hvis $\underline{\omega^2 = \frac{k}{m} + \frac{g}{R}}$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} + \frac{g}{R}} = \frac{1}{2\pi}\sqrt{\frac{7}{0{,}02} + \frac{9{,}81}{1}}\ Hz = 3{,}02\ Hz$$

$$T = \frac{1}{f} = \frac{1}{3{,}02}\ s = \underline{0{,}33\ s}$$

$v = v_{max}$ ved $t = T/4 = \frac{1}{4} \cdot 0.33\ s = 0.0825\ s$

$$B_0 = R \cdot \theta_0 = 1 \cdot 5 \cdot \frac{\pi}{180}\ m = 0{,}087\ m$$

$$v_{max} = R \cdot \left(\frac{d\theta}{dt}\right)_{t = T/4} = R \cdot \omega \cdot B_0 \cos\left(\omega \cdot \frac{T}{4} + \gamma\right)$$

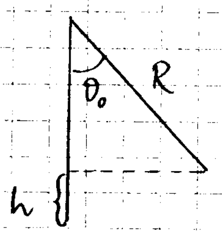
$$\theta_0 = B_0 = B_0 \sin\gamma \Rightarrow \gamma = \frac{\pi}{2} = 90^\circ$$

$$v_{max} = 1 \cdot \sqrt{\frac{k}{m} + \frac{g}{R}} \cdot 5 \cdot \frac{\pi}{180} \cdot \cos\left(\sqrt{\frac{k}{m} + \frac{g}{R}} \cdot 0.0825 + \frac{\pi}{2}\right) m/s = \underline{-1{,}66\ m/s}$$

Minustegnet angir retning mot venstre i fig

Alternativ metode – energibetraktning:

Ved $t = T/4$ vil den potensielle energien ved $t=0$ være gått over i kinetisk energi.



$$h = R(1-\cos\vartheta_0) = R\left(1-\left(1-2\sin^2\frac{\vartheta_0}{2}\right)\right)$$

$$\approx R\left(1-\left(1-2\left(\frac{\vartheta_0}{2}\right)^2\right)\right) = R\cdot\frac{\vartheta_0^2}{2}$$

$$\frac{1}{2}m v_{max}^2 = mgh + \frac{1}{2}k(R\vartheta_0)^2$$

$$\frac{1}{2}m v_{max}^2 = mgR\frac{\vartheta_0^2}{2} + \frac{1}{2}kR^2\vartheta_0^2$$

$$v_{max}^2 = \left(gR + \frac{k}{m}R^2\right)\vartheta_0^2 = R^2\left(\frac{k}{m}+\frac{g}{R}\right)\vartheta_0^2 = R^2\omega^2\vartheta_0^2$$

$$v_{max} = R\omega\vartheta_0 = 1\cdot\sqrt{\frac{7}{0.02}+\frac{9.81}{1}}\cdot 5\cdot\frac{\pi}{180}\ \mathrm{m/s} = \underline{1.66\ \mathrm{m/s}}$$

c) Så lenge vi kan anta at $\sin\vartheta \approx \vartheta$ vil ikke frekvensen og perioden bli påvirket av valget av ϑ_0 $\Longrightarrow T' = \underline{0.33\,s}$

$$v'_{max} = R\omega\vartheta_0 = 1\cdot\sqrt{\frac{7}{0.02}+\frac{9.81}{1}}\cdot 8\cdot\frac{\pi}{180}\ \mathrm{m/s} = \underline{2.65\ \mathrm{m/s}}$$

d) $$m\cdot R\cdot\frac{d^2\vartheta}{dt^2} = -kR\vartheta - mg\vartheta - bR\frac{d\vartheta}{dt}$$

$$\underline{\frac{d^2\vartheta}{dt^2} + \frac{b}{m}\frac{d\vartheta}{dt} + \left(\frac{k}{m}+\frac{g}{R}\right)\vartheta = 0}$$

$$\vartheta = Ce^{-\alpha t}\sin(\omega t+\delta)$$

$$\frac{d\vartheta}{dt} = -\alpha Ce^{-\alpha t}\sin(\omega t+\delta) + \omega Ce^{-\alpha t}\cos(\omega t+\delta)$$

$t=0: \quad \theta_0 = C \sin\delta$ (I)

$$\left(\frac{d\theta}{dt}\right)_{t=0} = -\alpha C \sin\delta + \omega C \cos\delta = 0 \quad \text{(II)}$$

Fra (II): $\tan\delta = \frac{\omega C}{\alpha C} = \underline{\frac{\omega}{\alpha}}$ q.e.d.

(I) innsatt i (II):

$$-\alpha\theta_0 + \omega C \cos\delta = 0$$

$$C\cos\delta = \frac{\alpha\theta_0}{\omega}$$

$$C^2(\sin^2\delta + \cos^2\delta) = C^2$$

$$\theta_0^2 + \frac{\alpha^2\theta_0^2}{\omega^2} = C^2$$

$$C^2 = \theta_0^2\left(\frac{\alpha^2}{\omega^2}+1\right) = \theta_0^2\frac{(\alpha^2+\omega^2)}{\omega^2} = \theta_0^2\frac{(\alpha^2+\omega_0^2-\alpha^2)}{\omega^2}$$

$\underline{C = \theta_0\frac{\omega_0}{\omega}}$ q.e.d.

e) Resterende amplitude pr syklus:

$$e^{-\alpha T} = e^{-\frac{b}{2m}T} \qquad T = \frac{2\pi}{\omega} = 2\pi\left(\frac{k}{m}+\frac{g}{l}-\frac{b^2}{4m^2}\right)^{-1/2}$$

Etter n sykluser er amplituden $\leq 50\%$ av startverdi

$$e^{-\frac{b}{2m}\cdot nT} \leq 0.5 \implies e^{\frac{b}{2m}nT} \geq 2$$

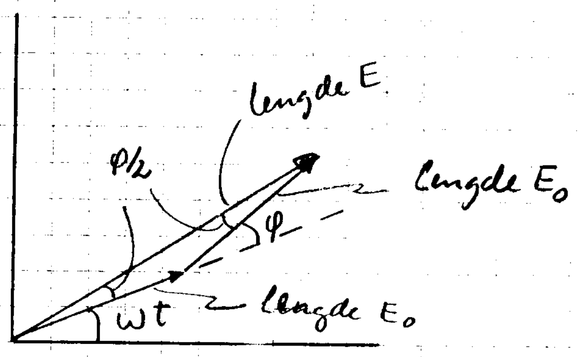
$$\frac{b}{2m}nT > \ln 2$$

$$n \geq \frac{2m\ln 2}{bT} = \frac{2\cdot 0.02\cdot\ln 2}{0.02}\sqrt{\frac{7}{0.02}+\frac{9.81}{1}-\frac{0.02^2}{4\cdot 0.02^2}} = 4.2$$

$\implies n = \underline{5}$

Oppgave 2

a) Viserdiagram:



Vi anvender utvidet Pythagoras på trekanten som dannes av de tre vektorene:

$$E^2 = E_0^2 + E_0^2 - 2E_0^2 \cos(\pi - \varphi)$$
$$= 2E_0^2 - 2E_0^2(\cos\pi\cos\varphi + \sin\pi\sin\varphi)$$

$$E^2 = 2E_0^2(1+\cos\varphi) = 2E_0^2\left(1+\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right)$$
$$= 2E_0^2\left(1+\cos^2\frac{\varphi}{2} - \left(1-\cos^2\frac{\varphi}{2}\right)\right) = 4E_0^2\cos^2\frac{\varphi}{2}$$

$|E| = 2E_0\cos\frac{\varphi}{2}$. Vektoren med denne lengden roterer med vinkelhastighet ω

$$\underline{E = E_\theta = 2E_0\cos\frac{\varphi}{2}\sin\left(\omega t + \frac{\varphi}{2}\right)}$$

Analytisk:

$$E = E_1 + E_2 = E_0\sin\omega t + E_0\sin(\omega t+\varphi) = E_0(\sin\omega t + \sin(\omega t+\varphi))$$

Nå er $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$$\Rightarrow E = E_0\cdot 2\sin\frac{\omega t+\omega t+\varphi}{2}\cos\frac{\omega t+\varphi-\omega t}{2}$$

$$\underline{E = 2E_0\sin\left(\omega t+\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}}$$

ω = vinkelfrekvens t = tid

φ = faseforskjell mellom de to strålene.

b) Konstruktiv interferens for

$\varphi = 2\pi n$ dvs for $\Delta L = n \cdot \lambda$

$$\frac{\varphi}{2\pi n} = \frac{\Delta L}{n\lambda} = \frac{d\sin\theta}{n\lambda} \Longrightarrow \varphi = \frac{2\pi d \sin\theta}{\lambda}$$

c) $$I = I(\lambda_1) + I(\lambda_2) = I_0\left[\cos^2\left(\frac{\pi d y}{L\cdot\lambda_1}\right) + \cos^2\left(\frac{\pi d y}{L\cdot\lambda_2}\right)\right]$$

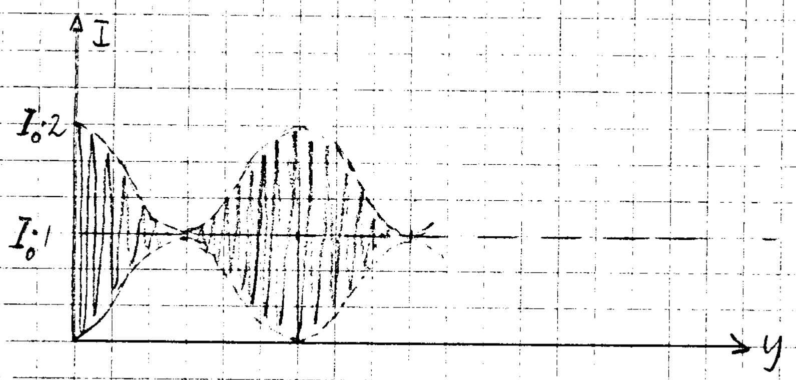
$$= \frac{I_0}{2}\left(\cos\left(\frac{2\pi d y}{L\lambda_1}\right) + 1 + \cos\left(\frac{2\pi d y}{L\lambda_2}\right) + 1\right)$$

$$= \frac{I_0}{2}\left(2 + 2\cos\left[\frac{2\pi d y}{2L}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\right]\cdot\cos\left[\frac{2\pi d y}{2L}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\right]\right)$$

$$= I_0\left(1 + \cos\left[\frac{\pi d y}{L}\frac{(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}\right]\cdot\cos\left[\frac{\pi d y}{L}\frac{(\lambda_2-\lambda_1)}{\lambda_1\lambda_2}\right]\right)$$

$$= I_0\left(1 + \cos\left(\frac{\pi d y\, 2\lambda_m}{L\cdot\lambda_m^2}\right)\cdot\cos\left(\frac{\pi d y\,\Delta\lambda}{L\lambda_m^2}\right)\right)$$

$$= \underline{I_0\left(1 + \cos\left(\frac{2\pi d y}{L\lambda_m}\right)\cdot\cos\left(\frac{2\pi d y\,\Delta\lambda}{2L\lambda_m^2}\right)\right)}$$



d) Stripene forsvinner første gang for

$$\cos\left(\frac{2\pi d\, y\, \Delta\lambda}{2L\, \lambda_m^2}\right) = 0$$

$$\frac{\pi d\, y\, \Delta\lambda}{L\, \lambda_m^2} = \pi/2$$

$$\underline{y = \frac{L \cdot \lambda_m^2}{2\, d\, \Delta\lambda}} = \frac{0.5\cdot(502.5\cdot 10^{-9})^2}{2\cdot 5\cdot 10^{-3}\cdot 5\cdot 10^{-9}}\ m = \underline{2.5\cdot 10^{-3}\ m = 2.5\ mm}$$

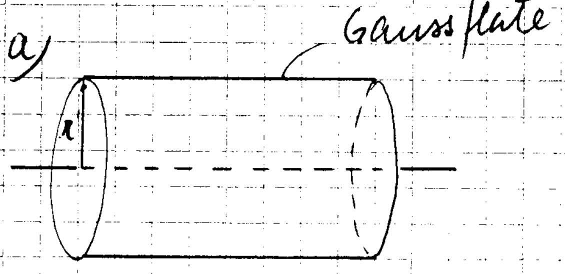
e) Bølgelengden i væsken er $\lambda_n = \frac{\lambda}{n}$

Stripeavstand $y' = \frac{L\cdot\lambda_n}{d} = \frac{L\cdot\lambda}{n d} = 33\cdot 10^{-6}\ m$

$$n = \frac{L\cdot\lambda}{d\cdot y'} = \frac{0.5\cdot 500\cdot 10^{-9}}{5\cdot 10^{-3}\cdot 33\cdot 10^{-6}} = \underline{1.52}$$

Oppgave 3

a)



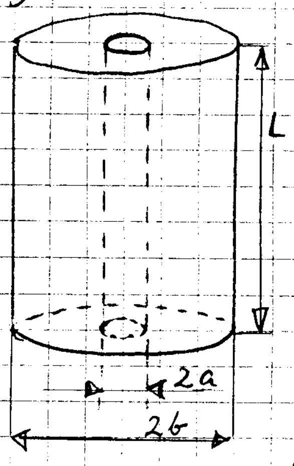
Gauss: $\oint \bar{E}\,d\bar{A} = \frac{Q}{\varepsilon_0}$

$$E \cdot 2\pi r L = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{2\pi\varepsilon_0 r L} \quad \text{rettet radielt}$$

$$\Rightarrow \bar{E} = \frac{Q}{2\pi\varepsilon_0 L}\frac{\hat{r}}{r} = \frac{Q}{2\pi\varepsilon_0 L}\frac{\vec{r}}{r^2}$$

b)



$$V_a - V_b = -\int_b^a \bar{E}\,d\bar{r} = V$$

$$V = \frac{-Q}{2\pi\varepsilon_0 L}\int_b^a \frac{1}{r}\,dr = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

Med dielektrikum i stedet for vakuum mellom tråden og sylinderen, må vi erstatte ε_0 med $\varepsilon = \kappa\varepsilon_0$, og får

$$C = \frac{2\pi\varepsilon_0 \kappa L}{\ln(b/a)} \quad \text{q.e.d.}$$

c) $V' = \frac{q}{C}$ ved et tidspkt. Tilfører enda en ladning dq:

$$dW = V'dq = \frac{q}{C}dq$$

$$W = \int dW = \int_0^Q \frac{q}{C}dq = \frac{1}{C}\cdot\frac{Q^2}{2}$$

$\Rightarrow U = \frac{Q^2}{2C}$ idet vi setter $U = 0$ for uladet kondensator

$$U = \frac{Q^2}{2C} = \frac{QV}{2} = \frac{CV^2}{2}$$

d) $$C = \frac{2\pi\varepsilon_0 \kappa L}{\ln(b/a)} = \frac{2\pi \cdot 8{,}85 \cdot 10^{-12} \cdot 80 \cdot 0{,}1}{\ln(2/1)} F = \underline{6{,}4 \cdot 10^{-10} F}$$

$$V = \frac{Q}{C} = \frac{1{,}0 \cdot 10^{-10}}{6{,}4 \cdot 10^{-10}} V = \underline{0{,}156 V}$$

$$U = \frac{1}{2} C V^2 = \frac{1}{2} \cdot 6{,}4 \cdot 10^{-10} \cdot 0{,}156^2 J = \underline{7{,}8 \cdot 10^{-12} J}$$

e) Det er samme spenning over de to delene av kondensatoren $\Rightarrow$ De to delene kan oppfattes som to separate kondensatorer som er koblet i parallell:

$$C = C_{fylt} + C_{luft} = \frac{2\pi\varepsilon_0 (z \cdot \kappa + L - z)}{\ln(b/a)}$$

$$= \frac{2\pi\varepsilon_0 (L + z(\kappa - 1))}{\ln(b/a)}$$

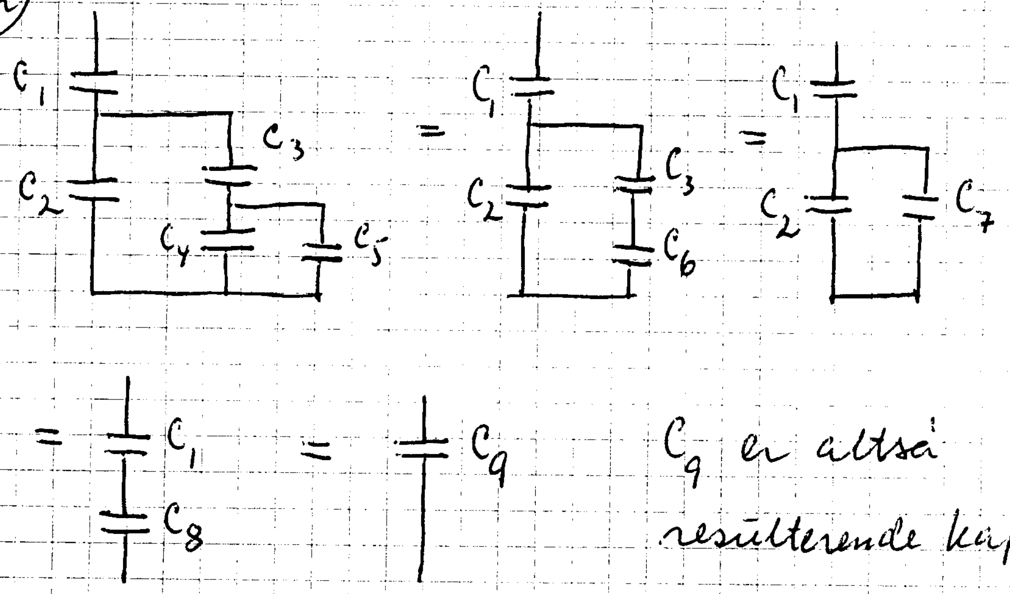
$$= \frac{2\pi \, 8{,}85 \cdot 10^{-12} (0{,}1 + 0{,}01 \cdot 79)}{\ln 2} F = \underline{7{,}1 \cdot 10^{-11} F}$$

$$V = \frac{Q}{C} = \frac{1{,}0 \cdot 10^{-10}}{7{,}1 \cdot 10^{-11}} V = \underline{1{,}4 V}$$

$$U = \frac{1}{2} C V^2 = \frac{1}{2} \cdot 7{,}1 \cdot 10^{-11} \cdot 1{,}4^2 J = \underline{7 \cdot 10^{-11} J}$$

Oppgave 4

a)



C_9 er altså resulterende kapasitans

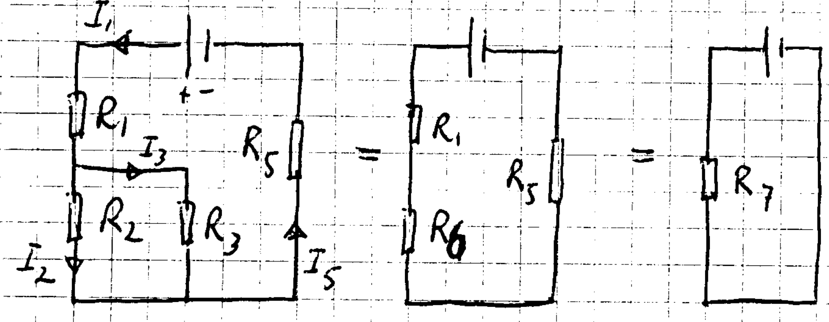
$$C_6 = C_4 + C_5 = (4+5)\mu F = 9\mu F$$

$$C_7 = \frac{C_3 \cdot C_6}{C_3 + C_6} = \frac{3 \cdot 9}{3+9}\mu F = 2{,}25\mu F$$

$$C_8 = C_2 + C_7 = (2 + 2{,}25)\mu F = 4{,}25\mu F$$

$$C_9 = \frac{C_1 \cdot C_8}{C_1 + C_8} = \frac{1 \cdot 4{,}25}{1 + 4{,}25}\mu F = \underline{0{,}81\mu F} = C$$

b) Steady state: Ingen strøm i kondensatornettet, dvs ikke spenningsfall over R_4.



$$R_6 = \frac{R_2 \cdot R_3}{R_2 + R_3} = \frac{20 \cdot 30}{20+30}\Omega = 12\,\Omega$$

$$R_7 = R_1 + R_5 + R_6 = (10+50+12)\,\Omega = 72\,\Omega$$

$$I_1 = I_5 = \frac{U}{R_7} = \frac{12}{72}\,A = \underline{0{,}17\,A}$$

$$V_6 = I_1 \cdot R_6 = 0{,}17 \cdot 12\,V = 2\,V$$

$$I_2 = \frac{V_6}{R_2} = \frac{2}{20}\,A = \underline{0{,}10\,A}$$

$$I_3 = I_1 - I_2 = (0{,}17 - 0{,}10)\,A = \underline{0{,}07\,A}$$

Retninger som angitt i figuren.

c) $Q_9 = C_9 \cdot V_6 = 0{,}81 \cdot 10^{-6} \cdot 2\,C = 1{,}62\,\mu C = Q$

$$Q_1 = Q_8 = Q_9 = \underline{1{,}62\,\mu C}$$

$$V_8' = \frac{Q_8}{C_8} = \frac{1{,}62 \cdot 10^{-6}}{4{,}25 \cdot 10^{-6}}\,V = 0{,}38\,V = V_7' = V_2'$$

der V_8', V_7' etc er spenninger over kondensatorer med tilsvarende indekser.

$$Q_2 = V_2' \cdot C_2 = 0{,}38 \cdot 2 \cdot 10^{-6}\,C = \underline{0{,}76\,\mu C}$$

$$Q_3 = Q_6 = Q_7 = V_7' \cdot C_7 = 0{,}38 \cdot 2{,}25 \cdot 10^{-6}\,C = \underline{0{,}86\,\mu C}$$

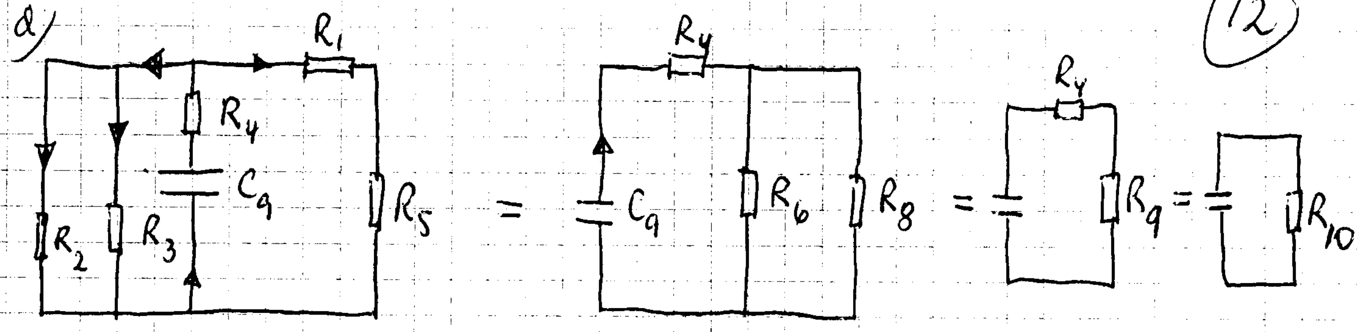
$$V_6' = \frac{Q_6}{C_6} = \frac{0{,}86 \cdot 10^{-6}}{9 \cdot 10^{-6}}\,V = 0{,}10\,V$$

$$Q_4 = V_6' \cdot C_4 = 0{,}10 \cdot 4 \cdot 10^{-6}\,C = \underline{0{,}4\,\mu C}$$

$$Q_5 = V_6' \cdot C_5 = 0{,}10 \cdot 5 \cdot 10^{-6}\,C = \underline{0{,}5\,\mu C}$$

At $Q_4 + Q_5 \neq Q_6$ skyldes avrundingsfeil.

d)



$R_6 = 12\,\Omega \qquad R_8 = R_1 + R_5 = (10+50)\,\Omega = 60\,\Omega$

$$R_9 = \frac{R_6 \cdot R_8}{R_6 + R_8} = \frac{12 \cdot 60}{12+60}\,\Omega = 10\,\Omega$$

$R_{10} = R_4 + R_9 = (40+10)\,\Omega = 50\,\Omega = R$

$$\frac{q}{C} - I \cdot R = 0 \qquad I = -\frac{dq}{dt}$$

$$\frac{dq}{q} = -\frac{dt}{RC} \Rightarrow \ln q = -\frac{t}{RC} + K'$$

$$q = K e^{-t/RC} = Q e^{-t/RC}$$

$$I = -\frac{dq}{dt} = \frac{Q}{RC} e^{-t/RC} = \frac{1{,}62 \cdot 10^{-6}}{50 \cdot 0{,}81 \cdot 10^{-6}} e^{-\frac{t}{50 \cdot 0{,}81 \cdot 10^{-6}}}$$

$\underline{I = 0{,}04\, e^{-\frac{t \cdot 10^6}{40{,}5}}\ \text{A}}$ når t måles i sek

Spenningsfallet over R_4 blir da

$$V_4 = I \cdot R_4 = 0{,}04 \cdot e^{-\frac{t \cdot 10^6}{40{,}5}} \cdot 40\ \text{V} = \underline{1{,}6 \cdot e^{-\frac{t \cdot 10^6}{40{,}5}}\ \text{V}}$$