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Solution to the exam in
SIF4072 CLASSICAL FIELD THEORY

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This solution consists of 8 pages.

Problem 1.

In this problem you shall consider some aspects of the model defined by the Liouville Lagrangian density

$$\mathcal{L} = (\partial_\mu \varphi)\,(\partial^\mu \varphi) - \kappa^2\, \mathrm{e}^{\lambda\varphi}, \tag{1}$$

where φ is a real field, and κ and λ are constants.

Comment:
Here a factor $\frac{1}{2}$ was unintentionally omitted: The convential normalization of the kinetic term is $\frac{1}{2}\,(\partial_\mu \varphi)\,(\partial^\mu \varphi)$ instead of $(\partial_\mu \varphi)\,(\partial^\mu \varphi)$.

a) Which physical dimensions must φ, κ and λ have for the action

$$S = \int \mathrm{d}t\, \mathrm{d}^d x\, \mathcal{L} \tag{2}$$

to have dimension $\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-1}$? Assume here that d is an arbitrary positive integer (the number of space dimensions).

The Lagrangian must have dimension $[\mathcal{L}]$ such that

$$\mathrm{s}\,\mathrm{m}^d\;[\mathcal{L}] = \mathrm{s}\,\mathrm{m}^{d-2}\;[\varphi]^2 = \mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-1}.$$

I.e.

$$[\varphi] = \mathrm{kg}^{1/2}\,\mathrm{m}^{2-d/2}\,\mathrm{s}^{-1}. \tag{3}$$

Further, the combination $\lambda\varphi$ must be dimensionless for the exponential of it to make physical sense,

$$[\lambda] = [\varphi]^{-1} = \mathrm{kg}^{-1/2}\,\mathrm{m}^{-2+d/2}\,\mathrm{s}. \tag{4}$$

Finally, since we must have $\left[\kappa^2\right] = [\mathcal{L}]$,

$$[\kappa] = \mathrm{m}^{-1}\;[\varphi] = \mathrm{kg}^{1/2}\,\mathrm{m}^{1-d/2}\,\mathrm{s}^{-1}. \tag{5}$$

b) Write down, or deduce, the field equations for this model.

This becomes the Liouville equation,

$$\Box\varphi = -\frac{1}{2}\lambda\,\kappa^2\,\mathrm{e}^{\lambda\varphi}, \tag{6}$$

where $\Box \equiv \partial_\mu \partial^\mu$ is the wave operator.

c) Assume that the field φ only depends on t and z, and show that

$$\varphi(t,z) = \sigma\ \log\left\{\frac{k\,A'(ct-z)B'(ct+z)}{[A(ct-z)+B(ct+z)]^2}\right\} \tag{7}$$

is solution of the field equations for suitable choice of constants σ and k. Here c is the speed of light. A and B are two general (twice differentiable) functions of one variable, and $'$ differentiation with respect to this variable.

Here (with dependence only on t and z) a standard, labor-saving technique is to factorize the wave operator as

$$\Box = (\partial_0 + \partial_z)\,(\partial_0 - \partial_z) = (\partial_0 - \partial_z)\,(\partial_0 + \partial_z)\,,$$

and utilize the fact that

$$(\partial_0 + \partial_z)\,A'(x^0 - z) = (\partial_0 - \partial_z)\,B'(x^0 + z) = 0.$$

Comment: This was expected to be common knowledge for physics students students at this level. It turned out not to be, but it should have been. Now *you* should remember this for the rest of your life! There was also given a slight hint of this, through the fact that A and B was only required to be twice differentiable.

We find

$$\begin{aligned}\Box\varphi &= -\,(\partial_0 + \partial_z)\,\frac{4\sigma\,A'(ct-z)}{A(ct-z)+B(ct+z)} \\ &= \frac{8\sigma\,A'(ct-z)\,B'(ct+z)}{[A(ct-z)+B(ct+z)]^2} = \frac{8\sigma}{k}\,\mathrm{e}^{\varphi/\sigma}.\end{aligned}$$

Thus, the Liouville equation is fulfilled when

$$\sigma = \frac{1}{\lambda},\quad k = -\frac{16}{\kappa^2\,\lambda^2}. \tag{8}$$

Comment: Simple, huh? With one honorable exception all more-or-less gave up on this, — but some only after heroic efforts. The honorable exception did it the hard way, which is by no means impossibly difficult.

A solution equivalent to this one was found 150 years ago: J. Liouville, *Sur l'équation aux différances partielles* $\frac{d^2\log\lambda}{du dv} \pm \frac{\lambda}{2a^2} = 0$, J. Math pures et appliquées **18**, 71–72 (1853). The Liouville equation is the simplest of a large class of equations which (in some sense) can be solved exactly in two space-time dimensions. It has grown up a small research industry around them. This includes some local contributions (Erling G. B. Hohler, *Relativistic Field Equations with Exponential Interactions*, NTH doktor ingeniøravhandling 1995:28).

d) Show that the action S is invariant under translations in time and space

$$\boldsymbol{T}_\nu : \varphi(x) \to \tilde{\varphi}(x;\varepsilon) = \varphi(x + \varepsilon\,e_\nu), \tag{9}$$

where e_ν is a unit vector in the ν-direction, $(e_\nu)^\mu = \delta^\mu_\nu = \eta^\mu_\nu$.

We find

$$\tilde{\mathcal{L}}(x) = \mathcal{L}(\tilde{\varphi}(x), \partial_\mu\tilde{\varphi}(x)) = \mathcal{L}(\varphi(x+\varepsilon\,e_\nu), \partial_\mu\varphi(x+\varepsilon\,e_\nu)) = \mathcal{L}(x+\varepsilon\,e_\nu).$$

Thus, by changing integration variables, $x \to \tilde{x} = x + \varepsilon\, e_\nu$, we find the action to be invariant,

$$\tilde{S} = \frac{1}{c}\int \mathrm{d}^{d+1}x\, \tilde{\mathcal{L}}(x) = \frac{1}{c}\int \mathrm{d}^{d+1}\tilde{x}\, \mathcal{L}(\tilde{x}) = S.$$

Use the Nöther procedure to find the corresponding conservation laws.

We find

$$\begin{aligned} \Delta\varphi(x) &= \frac{d}{d\varepsilon}\tilde{\varphi}(x;\varepsilon)\Big|_{\varepsilon=0} = \partial_\nu\varphi(x), \\ \Delta\mathcal{L}(x) &= \frac{d}{d\varepsilon}\tilde{\mathcal{L}}(x;\varepsilon)\Big|_{\varepsilon=0} = \eta^\mu_\nu\, \partial_\mu\mathcal{L}(x) = \partial_\mu\eta^\mu_\nu\, \mathcal{L}(x). \end{aligned}$$

Thus (using the general expression for the Nöther current given at the end of the exam set) the conserved quantity is the canonical energy-momentum tensor,

$$T^\mu_\nu = 2\,\partial^\mu\varphi\,\partial_\nu\varphi - \eta^\mu_\nu\,\mathcal{L} \tag{10}$$

e) Show that the field equation is invariant under scale transformations (dilatations)

$$\boldsymbol{D} : \varphi(x) \to \tilde{\varphi}(x;\varepsilon) = \varphi(\mathrm{e}^\varepsilon\, x) + \gamma\varepsilon \tag{11}$$

for a suitable choice of the constant γ.

Let $y = \mathrm{e}^\varepsilon\, x$. Then

$$\Box_x\tilde{\varphi}(x) = \Box_x\varphi(\mathrm{e}^\varepsilon\, x) = \mathrm{e}^{2\varepsilon}\,\Box_y\varphi(y) = -\frac{1}{2}\lambda\kappa^2\mathrm{e}^{2\varepsilon}\,\mathrm{e}^{\lambda\varphi(y)} = -\frac{1}{2}\lambda\kappa^2\mathrm{e}^{2\varepsilon}\,\mathrm{e}^{\lambda\tilde{\varphi}(x)-\lambda\gamma\varepsilon}.$$

From this we see that $\tilde{\varphi}$ solves the Liouville equation when φ solves the Liouville equation, and

$$\gamma = \frac{2}{\lambda}. \tag{12}$$

f) For a suitable number of space dimensions d the action (2) is also invariant under the scale transformation (11). Determine this value of d, and use the Nöther procedure to find the corresponding conservation law.

It follows from the previous point that

$$\tilde{\mathcal{L}}(x) = \mathrm{e}^{2\varepsilon}\,\mathcal{L}(\mathrm{e}^\varepsilon x)$$

Thus, by changing integration variables, $x \to \tilde{x} = \mathrm{e}^\varepsilon x$, i.e. $\mathrm{d}^{d+1}x = \mathrm{e}^{-(d+1)\varepsilon}\,\mathrm{d}^{d+1}\tilde{x}$, we find

$$\tilde{S} = \frac{1}{c}\int \mathrm{d}^{d+1}x\,\tilde{\mathcal{L}}(x) = \mathrm{e}^{(1-d)\varepsilon}\,\frac{1}{c}\int \mathrm{d}^{d+1}\tilde{x}\,\mathcal{L}(\tilde{x}) = \mathrm{e}^{(1-d)\varepsilon}\,S.$$

I.e., the action is invariant in one space dimension,

$$d = 1. \tag{13}$$

Comment: Here some candidates forgot to transform the t-integral, and thus found $d = 2$.

We find for $d=1$,

$$\Delta\varphi(x) = \frac{d}{d\varepsilon}\tilde{\varphi}(x;\varepsilon)\Big|_{\varepsilon=0} = x^\nu\,\partial_\nu\varphi(x) + \gamma,$$
$$\Delta\mathcal{L}(x) = \frac{d}{d\varepsilon}\tilde{\mathcal{L}}(x;\varepsilon)\Big|_{\varepsilon=0} = 2\mathcal{L}(x) + x^\mu\partial_\mu\mathcal{L}(x) = \partial_\mu\left(x^\mu\mathcal{L}(x)\right).$$

Thus (using the general expression for the Nöther current given at the end of the exam set) the conserved quantity is the dilatation current,

$$D^\mu = 2x^\nu\partial^\mu\varphi\,\partial_\nu\varphi + 2\gamma\partial^\mu\varphi - x^\mu\mathcal{L} = x^\nu T^\mu_\nu + 2\gamma\partial^\mu\varphi. \tag{14}$$

Problem 2.

The Lagrange function for relativistic point particles with mass m under constant acceleration g in the z-direction is (in some fixed coordinate system) given by

$$L = -mc^2\sqrt{1-(\boldsymbol{v}/c)^2} + mgz. \tag{15}$$

Comment: This problem is only a minor extension of problem **5f)**, given as exercise during the semester. In view of this some candidates did not do as well as expected on this problem.

a) Write down, or deduce, the equations of motion for such particles.

The Euler Lagrange equations become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{m\boldsymbol{v}}{\sqrt{1-(\boldsymbol{v}/c)^2}} = mg\,\hat{e}_z. \tag{16}$$

Comment: This is a situation where more work may not only be a waste of time, but actually harmful. It is not useful to explicitly carry out the last time differentiation in equation (16), since it is now written in a form which may immediately be integrated. However, if one still insists on performing the differentiation, the result is

$$\frac{m\boldsymbol{a}_\perp}{\left[1-(\boldsymbol{v}/c)^2\right]^{1/2}} + \frac{m\boldsymbol{a}_\parallel}{\left[1-(\boldsymbol{v}/c)^2\right]^{3/2}} = mg\,\hat{e}_z, \tag{17}$$

where $\boldsymbol{a}_\perp$ is the component of $\boldsymbol{a}\equiv\dot{\boldsymbol{v}}$ which is orthogonal to $\boldsymbol{v}$, and $\boldsymbol{a}_\parallel$ the component of $\boldsymbol{a}$ which is parallel to $\boldsymbol{v}$. The combinations $m\left[1-(\boldsymbol{v}/c)^2\right]^{-1/2}$ and $m\left[1-(\boldsymbol{v}/c)^2\right]^{-3/2}$ are sometimes (disgustingly) called transverse and parallel mass.

b) Two such particles start in rest at the time $t=0$, at respectively the positions $\boldsymbol{r}_1(0)=(0,0,0)$ og $\boldsymbol{r}_2(0)=(0,0,z_0)$. Find the paths $\boldsymbol{r}_k(t)$ $(k=1,2)$ by these two particles for $t\geq 0$.

The equations (16) can immediately be integrated once. Using the initial conditions this gives ($v_z=\dot{z}$)

$$\frac{\dot{z}/c}{\sqrt{1-(\dot{z}/c)^2}} = \frac{gt}{c},$$

and $\dot{x}=\dot{y}=0$. Solving for $\dot{z}$ we find

$$\dot{z} = \frac{gt}{\sqrt{1+(gt/c)^2}}.$$

Integrating once more, using the initial conditions and the integral given at the end of the exam set, gives

$$\begin{aligned} x_1(t) = y_1(t) = 0, \quad z_1(t) &= -\frac{c^2}{g} + \frac{c^2}{g}\sqrt{1+(gt/c)^2}, \\ x_2(t) = y_2(t) = 0, \quad z_2(t) &= -\frac{c^2}{g} + z_0 + \frac{c^2}{g}\sqrt{1+(gt/c)^2}. \end{aligned} \tag{18}$$

c) How long does it take, *measured by the eigentime of these particles*, to travel one million light years from the starting point (measured in the fixed coordinate system)?

Assume that $g = 10\,\mathrm{m\,s^{-2}}$, and set $c = 3\times 10^8\ nn; \mathrm{m\,s^{-1}}$. One million light years $\approx 0.947\times 10^{22}$ m.

The relation between elapsed eigentime τ and coordinate time t follows from the relation

$$\mathrm{d}\tau = \sqrt{1-(\boldsymbol{v}/c)^2}\,\mathrm{d}t = \frac{\mathrm{d}t}{\sqrt{1+(gt/c)^2}}.$$

This can be integrated with the help of the formula given at the end of the exam set

$$\tau = \frac{c}{g}\log\left[\frac{gt}{c} + \sqrt{1+\left(\frac{gt}{c}\right)^2}\right]. \tag{19}$$

We may trade the coordinate time t for the coordinate distance z travelled, using equation (18). The expression looks slightly simpler if we introduce the characteristic time

$$\tau_g = c/g = 3\cdot 10^7\ \mathrm{s} = 0.95\ \text{years},$$

and the characteristic distance

$$z_g = c\tau_g = c^2/g = 9\cdot 10^{15}\ \mathrm{m} = 0.95\ \text{lightyears}.$$

The result is

$$\tau = \tau_g\,\log\left[1+\frac{z+\sqrt{z^2+2z\,z_g}}{z_g}\right] \sim \begin{cases} \sqrt{2z/g} & \text{for } z \ll z_g, \\ \tau_g\,\log(2z/z_g) & \text{for } z \gg z_g. \end{cases} \tag{20}$$

For $z = 10^6$ lightyears this gives

$$\tau \approx 13.8\ \text{years} \approx 4.4\times 10^8\ \mathrm{s}. \tag{21}$$

d) At the time $t = 0$ a light signal is sent from particle 1, i.e. from the position $\boldsymbol{r}_1(0) = (0,0,0)$, in the direction of particle 2 (i.e. along the z-axis). If the distance z_0 is large, $z_0 \geq z_{\max}$, this light signal will never reach particle 2.

Determine the limiting value $z_{\max}$.

The light signal follows the line $z_\gamma(t) = ct$. This will cross the line

$$z_2(t) = z_0 - z_g + \sqrt{z_g^2 + (ct)^2}$$

when $z_0 < z_g$, and not cross it when $z_0 \geq z_g$. Thus

$$z_{\max} = z_g = 0.95\ \text{lightyears} = 9\cdot 10^{15}\ \mathrm{m}. \tag{22}$$

Problem 3.

The line element of a static, spherical symmetric geometry can be written in the form

$$\mathrm{d}s^2 = \mathrm{e}^{2a}\,(c\,\mathrm{d}t)^2 - \mathrm{e}^{2b}\,\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{23}$$

where a and b may be functions of$r$.

a) Write down the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ for this geometry in these coordinates.

The metric $g_{\mu\nu}$ can be read out directly from the line element. It is the diagonal matrix

$$g_{\mu\nu} = \mathrm{diag}\left(\mathrm{e}^{2a}, -\mathrm{e}^{2b}, -r^2, -r^2\sin^2\theta\right). \tag{24}$$

We find $g^{\mu\nu}$ by inverting this matrix,

$$g^{\mu\nu} = \mathrm{diag}\left(\mathrm{e}^{-2a}, -\mathrm{e}^{-2b}, -r^{-2}, -r^{-2}\sin^{-2}\theta\right). \tag{25}$$

The Einstein equations of gravity read

$$G^\mu_\nu = 8\pi\kappa T^\mu_\nu. \tag{26}$$

Assume he relevant (for this problem) components of the energy-momentum tensor takes the form

$$\begin{aligned} T^0_0 &= \begin{cases} \rho\,c^2 + \Lambda/\kappa, & 0 \le r \le r_s, \\ \Lambda/\kappa, & r_s < r, \end{cases} \\ T^1_1 &= \Lambda/\kappa, \end{aligned} \tag{27}$$

where ρ and Λ are positive constants with $\Lambda/\kappa \ll \rho\,c^2$. This corresponds to a star with constant mass density ρ and radius r_s in a universe with cosmological constant Λ.

b) Consider first the $(\mu,\nu) = (0,0)$ component of equation (26). Show that you by introducing

$$\mathrm{e}^{-2b(r)} = 1 - \frac{R(r)}{r} \tag{28}$$

can rewrite this equation into the form $R'(r) = f(r)$, where the right hand side $f(r)$ is explicitly known.

Inserting the explicit expression for G^0_0 we find the $(0,0)$ component of (26) to be

$$\mathrm{e}^{-2b}\left(\frac{2b'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\kappa T^0_0.$$

By differentiating (28), or from (28) itself, we find

$$\mathrm{e}^{-2b}\,\frac{2b'}{r} = \frac{R'}{r^2} - \frac{R}{r^3}, \qquad -\mathrm{e}^{-2b}\,\frac{1}{r^2} + \frac{1}{r^2} = \frac{R}{r^3}.$$

Thus, the equation becomes

$$R' = 8\pi\kappa r^2 T^0_0 = \begin{cases} 8\pi\kappa r^2\rho\,c^2 + 8\pi r^2\Lambda, & 0 \le r \le r_s, \\ 8\pi r^2\Lambda, & r_s < r. \end{cases} \tag{29}$$

c) Assume that $b(0) = 0$, and write down the explicit expression for $\mathrm{e}^{-2b(r)}$.

Since $b(0) = 0$ corresponds to $R(0) = 0$, we find

$$\mathrm{e}^{-2b(r)} = 1 - \frac{R(r)}{r} = \begin{cases} 1 - \frac{8\pi}{3}\kappa\rho\,c^2 r^2 - \frac{8\pi}{3}\Lambda r^2, & 0 \le r \le r_s, \\ 1 - \frac{8\pi}{3}\kappa\rho\,c^2 r_s^3\, r^{-1} - \frac{8\pi}{3}\Lambda r^2, & r_s < r. \end{cases} \tag{30}$$

Comment: Some candidates give the solution as $\mathrm{e}^{-2b(r)} = 1 - \frac{8\pi}{3}\kappa r^2 T^0_0$. This is not correct for $r > r_s$.

d) Show, by combining the $(\mu, \nu) = (0,0)$ and $(\mu, \nu) = (1,1)$ components of equation (26), that $a(r) + b(r)$ does not vary with r in the region $r > r_s$.

By subtracting the $(1,1)$ component of (26) from the $(0,0)$ component, we find

$$\frac{2}{r}\,\mathrm{e}^{-2b}\left(a' + b'\right) = 8\pi\kappa\left(T_0^0 - T_1^1\right). \tag{31}$$

The right hand side vanishes for $r > r_s$, since $T_0^0 = T_1^1$ in this region. Thus $a' + b' = 0$ in this region. I.e., $a + b$ does not vary with r. With a suitable choice of time coordinate we may choose $a = -b$ in this region.

e) The solutions for $a(r)$ and $b(r)$ breaks down when r becomes greater than a maximal radius $r_{\max}$. What do you think is the reason for that?

The main purpose of this question was to alert the candidate that something is fishy with the approach to cosmology taken in this problem, not to test explicit knowledge. Thus, all answers to this question will be considered with utmost generosity.

We note that there is a radius for which e^{2b} becomes infinite and e^{2a} becomes zero (taking $a = -b$). This is a signal that we are not using a globally valid coordinate system. One thing which may go wrong is that r becomes smaller again as we go further away from the origin, just like the cirumference of a constant altitude circle start to decrease again when we pass equator. Another problem is that our *a priory* assumption of a time independent geometry is highly questionable.

Some expressions of potential use:

The Euler Lagrange equations:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi^a\right)}=\frac{\partial \mathcal{L}}{\partial \varphi^a}. \tag{32}$$

the Nöther theorem:

$$J^\mu=\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi^a\right)} \Delta \varphi^a-M^\mu, \text{ when} \tag{33}$$

$$\Delta \mathcal{L}=\partial_\mu M^\mu, \quad \text{where } \Delta X \equiv \left.\frac{d}{d \varepsilon} \tilde{X}\right|_{\varepsilon=0} \quad \text{for } X \text{ equal } \varphi^a \text{ and } \mathcal{L}. \tag{34}$$

Two integrals:

$$\int \frac{\mathrm{d}t\, g t}{\sqrt{1+(g t / c)^2}}=\frac{c^2}{g} \sqrt{1+(g t / c)^2}, \quad \int_0^t \frac{\mathrm{d}t'}{\sqrt{1+\left(g t' / c\right)^2}}=\frac{c}{g} \log \left[\frac{g t}{c}+\sqrt{1+\left(\frac{g t}{c}\right)^2}\right] \tag{35}$$

Some relations from differential geometry:

$$\Gamma^\alpha{}_{\beta \gamma}=\frac{1}{2} g^{\alpha \mu}\left(g_{\mu \beta, \gamma}+g_{\mu \gamma, \beta}-g_{\beta \gamma, \mu}\right), \tag{36}$$

$$R^\alpha{}_{\beta \gamma \delta}=\Gamma^\alpha{}_{\beta \delta, \gamma}-\Gamma^\alpha{}_{\beta \gamma, \delta}+\Gamma^\alpha{}_{\mu \gamma} \Gamma^\mu{}_{\beta \delta}-\Gamma^\alpha{}_{\mu \delta} \Gamma^\mu{}_{\beta \gamma}, \tag{37}$$

$$R_{\beta \delta}=R^\alpha{}_{\beta \alpha \delta}, \tag{38}$$

$$G_{\beta \delta}=R_{\beta \delta}-\frac{1}{2} g_{\beta \delta} R. \tag{39}$$

The Einstein tensor for a spherical symmetric geometry:
With a spherical symmetric line element of the form

$$\mathrm{d}s^2=\mathrm{e}^{2a}(c\, \mathrm{d}t)^2-\mathrm{e}^{2b}\, \mathrm{d}r^2-r^2\left(\mathrm{d}\theta^2+\sin^2 \theta\, \mathrm{d}\varphi^2\right), \tag{40}$$

where a and b may depend on $x^0=c\,t$ and r, the non-vanishng components of the Einstein tensor is given by

$$\begin{aligned}
G_0^0 &= \mathrm{e}^{-2b}\left(\frac{2b'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}, \\
G_0^1 &= -\mathrm{e}^{-2b}\, \frac{2\dot{b}}{r}, \\
G_1^0 &= \mathrm{e}^{-2a}\, \frac{2\dot{b}}{r}, \\
G_1^1 &= -\mathrm{e}^{-2b}\left(\frac{2a'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}, \\
G_2^2 = G_3^3 &= -\mathrm{e}^{-2b}\left(a''+a'^2-a'\, b'+\frac{a'-b'}{r}\right)-\mathrm{e}^{-2a}\left(\dot{a}\, \dot{b}-\ddot{b}-\dot{b}^2\right),
\end{aligned} \tag{41}$$

where $\dot{}$ denotes differentiation with respect to x^0, and $'$ denotes differentiation with respect to r.