Section A

1.(a) Fermat's principle – principle of least time.

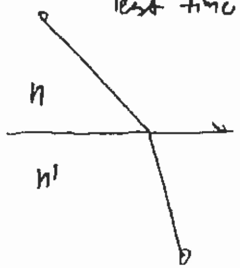
A ray follows the path from a point A to a point B that takes the least time

The more accurate/modern form of Fermat's principle is that $S(A,B) = \int_A^B n(s)\,ds$ – is optimal,

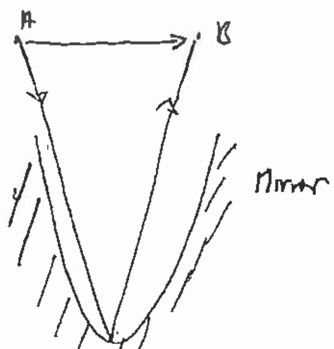
i.e. that $\delta S = \delta \int_A^B n(s)\,ds = 0$

Examples

least time

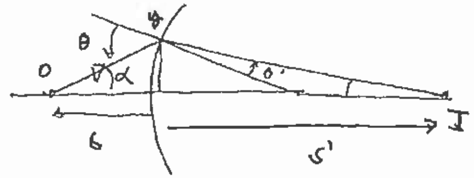


S – minimum



$\delta S = 0$!

1 b)



In deriving eqn. (A.1), several approximations are followed:

$n \sin\theta = n' \sin\theta' \rightarrow n\theta \simeq n'\theta'$,

i.e. the paraxial form of Snell's law.

Furthermore

$\alpha \approx \frac{y}{s}$

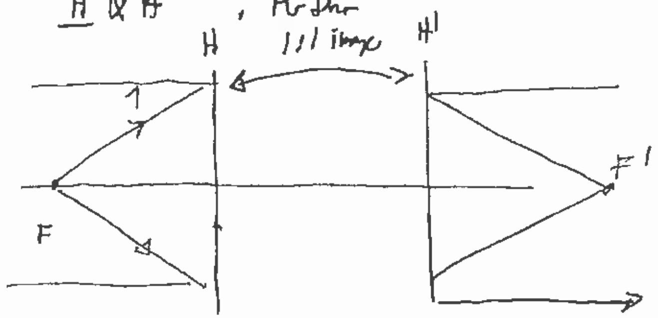
$\alpha' \approx \frac{y}{s'}$

} Gaussian first order optics only valid for rays close to the optic axis (z-axis)

Other assumptions:

- Spherical optics give perfect images
- no reflection losses, no scattering losses, no absorption losses
- no diffraction or size of lens/apertures.
-

1(c) Principal planes defined by a unit magnification between $\underline{H}$ & H', Further

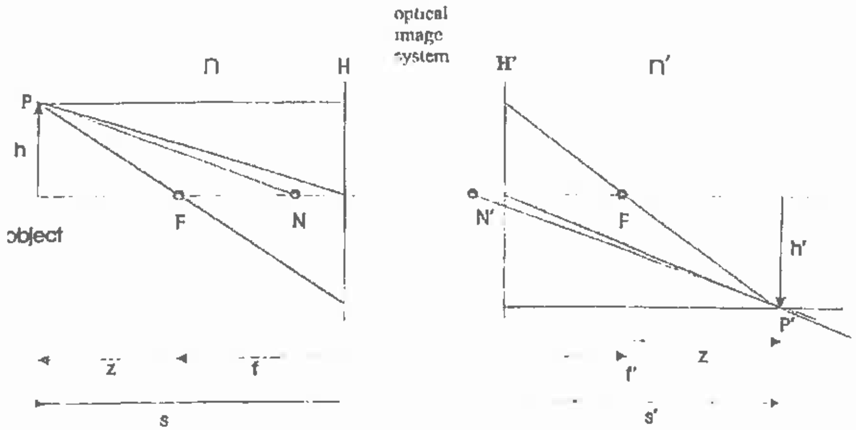


(1) rays from focal point F exits ∥ to optic axis

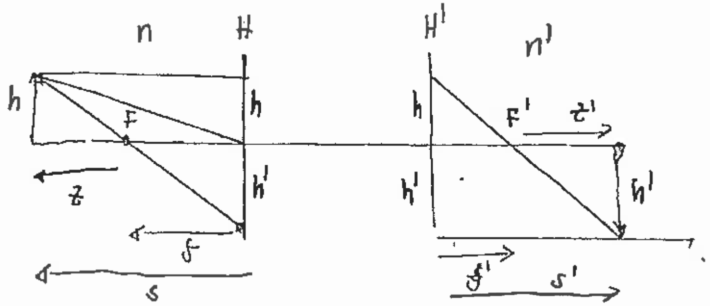
(2) rays through focal point F' ~~exits~~ input ~~far~~ ∥ to optic axis

Gaussian image eqn. $\frac{n}{s} + \frac{n'}{s'} = \frac{n'}{f}$ is valid w.r.t. principal planes H & H'

c)



1d)



From the figure it is evident that similar triangles on object side & image side may be used to derive the relationship between s', s & f'.

$$\frac{h}{z} = -\frac{h'}{f} \quad , \quad \frac{h}{f'} = -\frac{h'}{z'} \qquad (1)$$

hence

$$\frac{h'}{h} = -\frac{f}{z} = -\frac{z'}{f'} \quad \hat{=} \quad \underline{\text{transverse magnification } M_t} \qquad (2)$$

Newtonian image equation is given by:

$$ff' = zz' \qquad (3)$$

, in accordance with the appendix.

To obtain the Gaussian image eqn.: $\frac{n}{s} + \frac{n'}{s'} = \frac{n'}{f'}$ (4) see Appendix, we set

$$z = (s-f) \quad \& \quad z' = (s'-f') \qquad (5)$$

and insert into eqn. (3)

$$\Rightarrow \cancel{ff'} = (s-f)(s'-f') = ss' - fs' - sf' + ff' \; (zz')$$

$$\Rightarrow ss' = fs' + sf'$$

$$1 = \frac{f}{s} + \frac{f'}{s'} \qquad (6)$$

now we use that $\frac{n}{f} = \frac{n'}{f'}$, $\Rightarrow f = \frac{nf'}{n'}$ (7)

{ the latter equation (7) comes from the definition of the principal planes and that repeated application of many refracting surfaces always gives: $M_t \cdot M_\alpha = \frac{n}{n'}$

with $M_t = 1$ between principal planes, we have that

$$M_\alpha = \frac{n}{n'} = \frac{\alpha'}{\alpha}$$

$$\alpha n = n'\alpha' \quad \&$$

$\cancel{\alpha = \frac{h'}{f}}$

$\cancel{\alpha' = -\frac{h}{f'}}$

hence $\frac{n}{f} = \frac{n'}{f'}$

$$-\frac{FF_1}{f} = \alpha$$

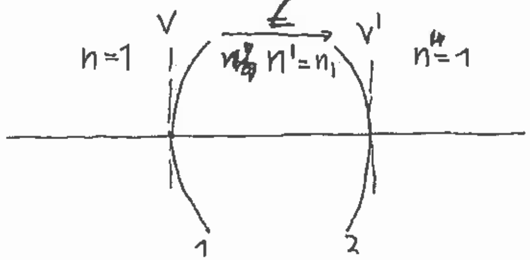
$$-\frac{H'F_1'}{f'} = \alpha' = -\frac{FF_1}{f'}$$

$$\Rightarrow 1 = \frac{nf'}{n's} + \frac{f'}{s'}$$

$$\Rightarrow \frac{n'}{s'} + \frac{n}{s} = \frac{n'}{f'}$$

A2.a/ Prove that for thin lens :

$$P = \frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$



produce system transfer matrix between V & V'

surface 2

$$M = \begin{bmatrix} 1 & 0 \\ \frac{n-n'}{n'} & \frac{n}{n'=1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ +C_2 & n_1 \end{bmatrix}$$

$$C_2 = \left(\frac{n-n'}{Rn'}\right) = \left(\frac{n_1-1}{R_2}\right)$$

transfer matrix $\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$

surface 1

$$\begin{bmatrix} 1 & 0 \\ C_1 & \frac{n}{n'} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C_1 & \frac{1}{n_1} \end{bmatrix}$$

$$C_1 = \left(\frac{1-n_1}{R_1 n_1}\right)$$

$$M_{VV'} = \begin{bmatrix} 1 & 0 \\ C_1 & \frac{1}{n_1} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C_2 & n_1 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ C_1 & \frac{1}{n_1} \end{bmatrix} \begin{bmatrix} 1 + C_2 L & L n_1 \\ C_2 & n_1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 + C_2 L & L n_1 \\ (1 + C_2 L) C_1 + \frac{C_2}{n_1} & L n_1 C_1 + 1 \end{bmatrix}$$

now for the approximation, it is known that the refractive Power is given by

$$P = \frac{1}{f'} = -C \qquad \text{, see table 18-2,}$$

hence

$$P = -(1 + C_2 L) C_1 + \frac{C_2}{n_1} \quad , L \to 0 \text{ for thin lens}$$

$$\Rightarrow -P = C_1 + \frac{C_2}{n_1} = \left(\frac{1-n_1}{R_1 n_1}\right) + \left(\frac{n_1 - 1)/1}{R_2/n_1}\right)$$

$$= \left(\frac{n_1-1}{n_1}\right)\left(-\frac{1}{r_1} + \frac{1}{r_2}\right), \quad P = \left(\frac{n_1-1}{n_1}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \text{ qtb}$$

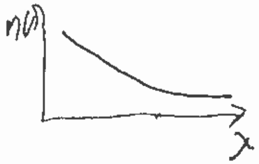
b) It is observed that

$$P = \frac{1}{f} = (n(\lambda)-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

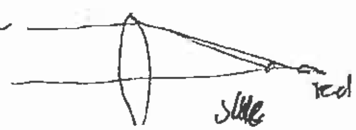
Dispersion means that $n(\lambda)$ has a wavelength dependence.

Therefore $P(\lambda) = \frac{1}{f(\lambda)}$

Typically, $n(\lambda)$ of glasses vary from $n \sim 1.5$ to 1.51 over a few ~~two~~ hundred nm's, and causes light of different wavelengths (colors) to have a different focal point. This is ~~called~~ denoted chromatic aberration.



example



2 c) This question is similar to question (a), set up

$$M_{f'} = M_{L_2} M_{trans} M_{L_1}$$

$$= \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix}$$

lenses in air

$$\begin{bmatrix} 1-LP_1 & L \\ -P_1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1-LP_1 & L \\ -P_2(1-LP_1)-P_1 & -P_2L+1 \end{bmatrix}$$

Using again the results from table 13-2.

then $\underline{\underline{P' = P_1 + P_2 - LP_1P_2}}$, where P' denotes total refractive power on image side

or $\underline{\frac{1}{f'} = \frac{1}{f_1} + \frac{1}{f_2} - L\frac{1}{f_1}\frac{1}{f_2}}$

d) $p = P_1 + P_2 - L P_1 P_2$

In order for the system to be overall achromatic, then we simply require that: (see e.g. Pedrotti p. 76.)
(see It's learning)

$$\frac{\partial P}{\partial \lambda} = \frac{\partial P}{\partial n}\left(\frac{\partial n}{\partial \lambda}\right) = 0$$

$$\frac{\partial P_1}{\partial \lambda} = \left(\frac{1}{R_{11}} - \frac{1}{R_{12}}\right)\cdot\frac{\partial n}{d\lambda} = \frac{1}{(n_1-1)}\, P_1 \left(\frac{\partial n}{\partial \lambda}\right)$$

$$\frac{\partial P_2}{\partial \lambda} = \frac{1}{(n-1)}\, P_2 \frac{\partial n}{d\lambda}$$

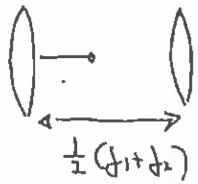
$$\Rightarrow \frac{\partial P}{\partial n} = \frac{\partial P_1}{\partial n} + \frac{\partial P_2}{\partial n} - L\left(\left(\frac{\partial P_1}{\partial n}\right) P_2 + \frac{\partial P_2}{\partial n} P_1\right)$$

$$= \frac{P_1}{(n-1)} + \frac{P_2}{(n-1)} - \frac{L}{(n-1)}\left(P_1 P_2 + P_2 P_1\right) = 0$$

$$L = \frac{P_1 + P_2}{2 P_1 P_2} = \frac{1}{2}\left[\frac{1}{P_2} + \frac{1}{P_1}\right]$$

$$= \underline{\underline{\frac{1}{2}\left(f_2 + f_1\right)}}$$

(contd.) The system is thus achromatic if the two lenses are separated by $\frac{1}{2}(f_1 + f_2)$



We note that $P^{s} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{2}(f_1 + f_2)\frac{1}{f_1}\frac{1}{f_2}$

$$= \frac{1}{2}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) = \frac{1}{2}(P_1 + P_2)$$

If both lenses are positive, then we have a normal imaging system, but we recognise the configuration found from the

- Huygens eye piece

  and the

- Ramsden eye piece.

The latter are used in telescopes & microscopes eye pieces.

See Pedrotti page 77.

e) Achromatic lens : see It's learning - posted notes, Pedrotti p. 455)

$P = P_1 + P_2 - L P_1 P_2$, $L \to 0$ for compound lens

$$\frac{\partial P}{\partial \lambda} = \frac{P_1}{(n_1-1)} \frac{dn_1}{d\lambda} + \frac{P_2}{(n_2-1)} \frac{dn_2}{d\lambda} = 0$$

$$\Rightarrow \frac{P_1}{(n_1-1)} \frac{dn_1}{d\lambda} = \frac{-P_2}{(n_2-1)} \frac{dn_2}{d\lambda}$$

$(n_1-1)(n_2-1) > 0$ always for transparent glass in the visible

$\Rightarrow$ P_2 must have opposite power of P_1

i.e. $\operatorname{sign}(f_1) = -\operatorname{sign}(f_2)$

For $P = (P_1 + P_2) > 0$, then

$P_1 > |P_2|$ if $P_1 > 0$

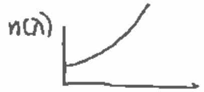
or $P_1 = (P - P_2) \Rightarrow \frac{(P-P_2)}{(n_1-1)} \frac{dn_1}{d\lambda} = -\frac{P_2}{(n_2-1)} \frac{dn_2}{d\lambda}$

Let $K_1 = \frac{1}{(n_1-1)} \frac{dn_1}{d\lambda}$ and $K_2 = \frac{1}{(n_2-1)} \frac{dn_2}{d\lambda}$

$\Rightarrow (P - P_2) K_1 = -P_2 K_2$

P_1 +ve, P +ve $\Big\}$ $P_2 = \frac{P}{(K_1 - K_2)} < 0$

hence $K_2 > K_1$

For a dense dispersive material 

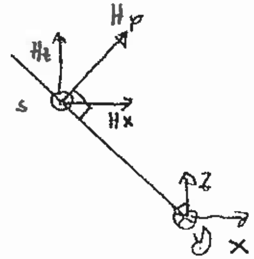
$\frac{dn}{d\lambda}$ is large

Hence high dispersion, $-P_2$, -ve focal length (typically crown)

low dispersion (typically flint)

$+P_1$, + focal length

B4a) $E(\vec{r},t) = E_s(\vec{r},t)\,\hat{e}_s = E_{0s}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$



Use Faradays law (Appendix, Maxwells eqns.): $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Assume time harmonic solutions, $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

then $\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$.

From material eqns. (Appendix) or constitutive equations

$\mu_0 \vec{H} = \vec{B}$, where we assume $\mu = 1$ in optics.

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & -E_s & 0 \end{vmatrix}$$

where $\partial_x \equiv \frac{\partial}{\partial x}$, $\partial_y \equiv \frac{\partial}{\partial y}$, $\partial_z \equiv \frac{\partial}{\partial z}$, $E_s = E_s(\vec{r},t)$

$$\vec{\nabla}\times\vec{E} = \hat{x}\frac{\partial}{\partial z}E_s - \hat{y}\cdot 0 + \hat{z}\left(-\frac{\partial}{\partial x}E_s\right)$$

$$= \hat{x}\, i k_z E_s - \hat{z}\, i k_x E_s$$

where we have used that $\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$

hence $i k_z E_s = -i\omega B_{xp} = -i\omega\mu_0 H_{xp}$

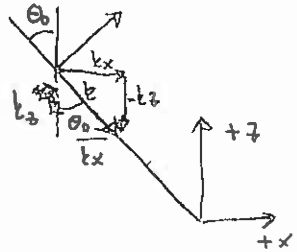
$-i k_x E_s = -i\omega B_{zp} = -i\omega\mu_0 H_{zp}$

$0 = -i\omega B_{yp}$

Hence $H_{xp} = -\frac{1}{\mu_0}\frac{k_z}{\omega}E_s$

$H_{zp} = \frac{1}{\mu_0}\frac{k_x}{\omega}E_s$

$H_{yp} = 0$



$\sin\theta_0 = \frac{k_x}{k}$

$\cos\theta_0 = -\frac{k_z}{k}$

$k_x = k\sin\theta_0$

$k_z = -k\cos\theta_0$

$\Rightarrow$ $H_{xp} = \frac{1}{\mu_0}\frac{k}{\omega}\sin\theta_0\, E_s$

$H_{yp} = 0$

$H_{zp} = \frac{1}{\mu_0}\frac{k}{\omega}\cos\theta_0\, E_s$

$\vec{H}_p(\vec{r},t) = \frac{1}{\mu_0}\frac{k}{\omega}\left[\sin\theta_0\,\hat{x} + \cos\theta_0\,\hat{z}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

also note that

$$|\vec{H}_p| = H_{0p} = \frac{1}{\mu_0}\frac{1}{v}E_{0s}$$

and that $\vec{H}_p(\vec{r},t) = \frac{1}{\mu_0}\frac{k}{\omega}\left(\hat{s} \times \vec{E}_s(\vec{r},t)\right)$

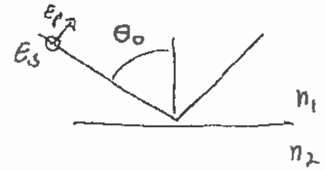
or $\vec{B}_p = \frac{1}{v}(\hat{s} \times \vec{E}_s)$

where $\hat{s} = \frac{\vec{k}}{k} = (\sin\theta_0, 0, -\cos\theta_0)$

i.e. $(\vec{H}_p \perp \vec{E}_s \perp \vec{k})$

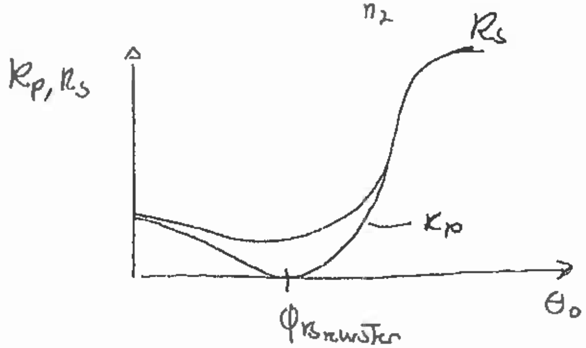
B1b)

i) Polarisation by reflection from a plane isotropic surface; at the Brewster angle.



e.g. Air

water

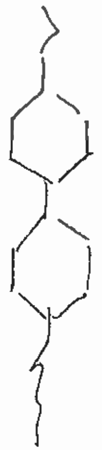


at $\theta_0 = \phi_{Brewster}$, then $|r_p|^2 = R_p = 0$

hence only s-polarised light is reflected, i.e. linearly polarised light

$$\tan(\phi_{Brewster}) = \frac{n_1}{n_2}$$

ii) Oriented polymer chains with an absorption that depends on the orientation of the electric field.
Also commonly known as "dichroic" polarizers:
camera filters, sunglasses etc.

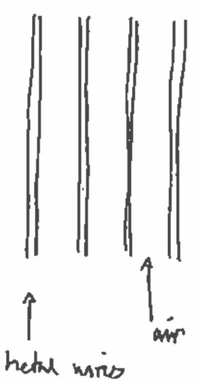


e.g.

(absorb light) when E-field ∥ to chain

~~transmit~~ transmit light when E field ⊥ to chain

iii) Metallic wire-grid polarizers



absorb light when field ∥ to wires (conduction electrons oscillate with field and works as metallic reflector)

transmit light when E-field ⊥ to wires

iv) Prism-cube like polarizers composed of isotropic and/or anisotropic medium with different crystal orientation. Also known as birefringent polarizers (Hecht)

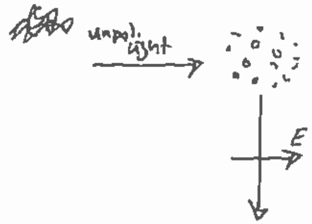
E.g. Glan Thompson polariser

Rochon Polarizer

Wollaston prism

See Pedrotti page 362 – 363, Figure 15-15; 15-16. See Hecht page 344, 345

v) Polarisation by scattering from particles



← Sub-wavelength particles, e.g. air

linearly pol. light

This is why you get better contrast looking at the sky with polarised "shades" (sunglasses) c)

retardance $\delta = \delta_e - \delta_o$

$= \frac{2\pi}{\lambda}(n_e - n_o)d$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$$

$\delta = \pi \Rightarrow$ Half wave plate

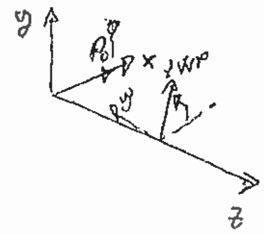
$\delta = \frac{\pi}{2} \Rightarrow$ QW plate

$\delta_{HW} = \pi = \frac{2\pi}{\lambda}(n_e - n_o)d$

$d = \frac{\lambda}{2(n_e - n_o)} = 3\times10^4$ nm $= \underline{30\ \mu m}$

δ_{QW} $d = \frac{\lambda}{4(n_e - n_o)} = \underline{15\ \mu m}$

M. d)



T_{WP} in x', y' system $= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$

Outgoing Jones vector in x', y' system

$$\bar{J}_{x',y'} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix} \bar{\bar{R}}(45) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$J_{x,y} = \bar{\bar{R}}(-45)\, \bar{\bar{T}}_{x',y'}\, R(45) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

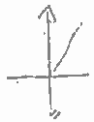
From Appendix

$$R(45) = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$R(-45) = \begin{bmatrix} \frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$\delta = \frac{3\pi}{2}$, which is probably going to work as a QW plate:

$$e^{i\delta} = \cos\frac{3\pi}{2} + i\sin\left(\frac{3\pi}{2}\right)$$



$$= 0 - i$$

$$J_{x,y}(\delta) = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{i\delta}\end{bmatrix}\underbrace{\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}}_{\begin{bmatrix}1\\ 1\end{bmatrix}}$$

$$\begin{bmatrix}1 & 0\\ 0 & e^{i\delta}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}1\\ e^{i\delta}\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 + e^{i\delta}\\ -1 + e^{i\delta}\end{bmatrix} = \cancel{\;}$$

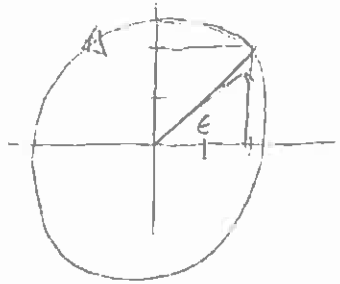
$$J_{x,y}\left(\delta = \frac{3\pi}{2}\right) = \frac{1}{2}\begin{bmatrix}1 + -i\\ -1 - i\end{bmatrix} = \frac{1}{2}(1-i)\begin{bmatrix}1\\ \frac{-1-i}{1-i}\end{bmatrix}$$

$$\frac{(-1-i)}{1-i} = \frac{(-1-i)(1+i)}{(1-i)(1+i)} = \frac{-1-i-i+1}{1^2+1^2}$$

$$= -\frac{2i}{2} = \underline{-i}$$

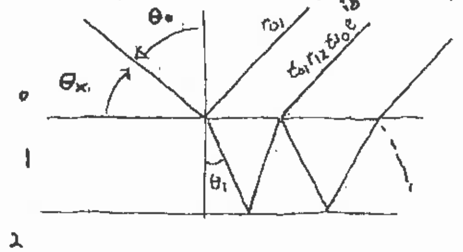
$$J_{x,y} = \begin{bmatrix}1\\ -i\end{bmatrix} \quad \text{so} \quad = \begin{bmatrix}1\\ i\tan\epsilon\end{bmatrix}$$

$$\tan\epsilon = \frac{E_{0y}}{E_{0x}} = -1$$



using the convention in the appendix, this is a left circular polarisation state!

B3a) Analyze ~~the~~ reflection from a thin film on a substrate, i.e. an amplitude splitting interferometer.



The reflection coefficients can be written as

$$r_{012} = \frac{E_{\text{reflected field}}}{E_{\text{incident field}}}$$

$$= r_{01} + t_{01}t_{10}r_{12}e^{i\delta} + t_{01}t_{10}r_{12}r_{10}r_{12}e^{i2\delta} + \dots$$

$$(1) \qquad = r_{01} + t_{01}t_{10}r_{12}e^{i\delta}\left[1 + \sum_{m=1}^{N}\left(r_{10}r_{12}e^{i\delta}\right)^{m}\right]$$

where $\delta = \frac{4\pi}{\lambda} d_{film}\, n_{film} \cos\theta_{film}$

B3b) From the series of multiple reflections, it is observed from eqn. B3a (1) that for every

$\delta = 0, 2\pi, 4\pi \dots \quad r_{012} = r_{02}$, since

$d_{film} = 0 \iff \delta = 0 \quad \left(e^{i\frac{4\pi}{\lambda} n_f \cos\theta_f d_f} = e^{i\delta} \right)$

As an approximation, we will estimate the thickness of the film between two maxima

$\delta_{i+1} - \delta_i = 2\pi$, corresponding to

$$\frac{4\pi d_{film}}{\lambda} n_{film} \left(\cos\theta_{film_{i+1}} - \cos\theta_{film_i} \right) = 2\pi$$

now $\cos(90 - \theta_i) = \sin\theta_x$, while

$$\cos\theta_{film} = \left(1 - \frac{n_0^2}{n_f^2}\sin^2\theta_0\right)^{\frac{1}{2}} = \left(1 - \frac{n_0^2}{n_f^2}\cos^2\theta_x\right)^{\frac{1}{2}}$$

use Snell's law $n_0 \sin\theta_0 = n_f \sin\theta_f$

$$\left(1 - \frac{n_0}{n_f}\sin\theta_0\right)^{\frac{1}{2}} = \cos\theta_{film}$$

$$\cos\theta_{film} \approx \left(1 - \frac{1}{n_f^2}(1-\theta_x)^2\right)^{\frac{1}{2}} \approx \left(1 - \frac{1}{n_f^2}(1 - 2\theta_x)\right)^{\frac{1}{2}}$$

$$\approx \sqrt{\frac{n_f^2 - 1}{n_f^2}}\left(1 - 2\theta_x\right)^{\frac{1}{2}} \approx \sqrt{\frac{n_f^2-1}{n_f^2}}\left(1 - \theta_x\right)$$

$$\left(\cos\theta_{f\,i+1} - \cos\theta_{f\,i}\right) \approx \theta_{x\,i} - \theta_{x\,i+1}$$ (but with sign here)

$$\frac{4\pi\, d_{film}}{\lambda}\, n_{film}\left(\theta_{x_{i+1}} - \theta_{x_i}\right) \approx 2\pi$$

$$d_{film} \approx \frac{\lambda}{d_{film}\, n_{film}} \cdot \frac{1}{\left(\theta_{x_{i+1}} - \theta_{x_i}\right)}$$

$$d_{film} \approx \frac{1.5406\ \text{Å}}{2\cdot\left(1-1.13\times10^{-5}\right)\left(4.8-3.9\right)\left(\frac{\pi}{180}\right)}$$

$$\approx 49\ \text{Å}$$

~~What is~~

More exact solution can be found from setting into :

$$d_{film} = \frac{\lambda}{2\, n_{film} \cdot \left[\left(1 - \frac{1}{n_{film}^2}\cos^2\theta_{x_i}\right)^{\frac{1}{2}} - \left(1 - \frac{1}{n_{film}^2}\cos^2\theta_{x_{i+1}}\right)^{\frac{1}{2}}\right]}$$

$$= \frac{\lambda}{2\left[\left(n_f^2 - \cos^2\theta_{x_i}\right)^{\frac{1}{2}} - \left(n_f^2 - \cos^2\theta_{x_{i+1}}\right)^{\frac{1}{2}}\right]}$$

B3c) We observe that $n_{film} < n_{air}$

(note that absorption of the order $k \sim 10^{-7}$ has been neglected)

$\Rightarrow$ total internal reflection in air

This happens for

$$n_{air} \sin\theta_{air} = n_f \sin\theta_f \overset{\theta_f = 90^\circ}{=} n_f$$

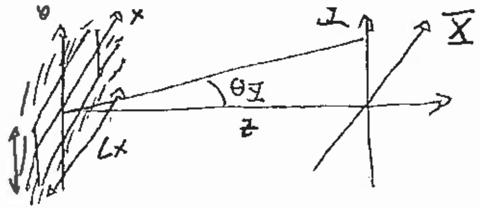
$$\sin\theta_{air}^{critical} = \frac{n_f}{n_{air}}$$

$$\theta_{air}^{critical} = \sin^{-1}\left(\frac{n_f}{n_{air}}\right) \approx \frac{n_f}{n_{air}}$$

$$\theta_x = 90 - \frac{n_f}{n_{air}} \cdot \frac{180}{\pi}$$

$\theta_x^{cutoff/critical} = 0.2724^\circ$, in good agreement with the figure.

13.4 a)



Fraunhofer diffraction integral is given by the appendix to be:

(1) $$U(\bar{X},\bar{Y},z) = \frac{U_0^-(x,y,0)}{i\lambda z} e^{ikz} e^{ik\frac{(\bar{X}^2+\bar{Y}^2)}{2z}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t(x,y)\, e^{-i(k_{\bar{X}}x+k_{\bar{Y}}y)}\, dx\, dy$$

where we have introduced the transmission function for the aperture $t(x,y)$ defined by:

$$t(x,y) = \frac{U_0^+(x,y,0)}{U_0^-(x,y,0)} \qquad (2)$$

where $U_0^-(x,y,0)$ is the field amplitude incident on the aperture,

(3) $$t(x,y) = \mathrm{rect}\left(\frac{x}{L_x},\frac{y}{L_y}\right) = \begin{cases} 1 & |x|<\frac{L_x}{2} \text{ and } |y|<\frac{L_y}{2} \\ 0 & \text{elsewise} \end{cases}$$

Eqn. (1) may be observed to be a Fourier transform times a prefactor, giving

$$U(\bar{X},\bar{Y},z) = \frac{U_0^-(x,y,0)}{i\lambda z} e^{ikz} e^{ik\frac{(\bar{X}^2+\bar{Y}^2)}{2z}} \mathcal{F}\{t(x,y)\}$$

where $\mathcal{F}\{t(x,y)\} = \mathcal{F}\left\{\mathrm{rect}\left(\frac{x}{L_x},\frac{y}{L_y}\right)\right\}$

$$\mathrm{rect}\left(\frac{x}{L_x},\frac{y}{L_y}\right) = \mathrm{rect}\left(\frac{x}{L_x}\right)\cdot \mathrm{rect}\left(\frac{y}{L_y}\right)$$, (separable function)

From the appendix we find

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{x}{L_x}\right)\right\} = L_x \,\mathrm{sinc}\left(\frac{k_{\bar{X}} L_x}{2}\right)$$

recall definition of Fourier transform:

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{x}{L_x}\right)\right\} \cancel{\iint} = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{x}{L_x}\right) e^{-ik_{\bar{X}}x}\, dx$$

Fourier space variable

The Fourier space variables $k_{\bar{X}} \simeq \frac{k\bar{X}}{z}$, $k_{\bar{Y}} \simeq \frac{k\bar{Y}}{z}$, normally.

Hence

$$\mathcal{F}\{t(x,y)\} = L_x L_y \,\mathrm{sinc}\left(\frac{k_{\bar{X}}L_x}{2}\right) \mathrm{sinc}\left(\frac{k_{\bar{Y}}L_y}{2}\right)$$

The intensity, or irradiance, is given by:

$$\langle I \rangle = \frac{1}{2}\epsilon_0 c\, |U(\bar{X},\bar{Y},z)|^2 = \frac{1}{2}\epsilon_0 c\, U(\bar{X},\bar{Y},z)\cdot U(\bar{X},\bar{Y},z)^*$$

where * denotes the complex conjugate.

Hence the intensity is given as

$$\langle I \rangle (\bar{X}, \bar{Y}, z) = \frac{I_0}{\lambda^2 z^2} L_x^2 L_y^2 \operatorname{sinc}^2\left(\frac{k\bar{X} L_x}{2}\right) \operatorname{sinc}^2\left(\frac{k\bar{Y} L_y}{2}\right),$$

where $I_0 = (U_0^-(x,y,0))^2 \cdot \frac{1}{2}\epsilon_0 c$

3.4b) The validity of the Fraunhofer approximation, with respect to the Fresnel approximation, is easily seen by observing the difference between the two, see Appendix. We see that if

$$(+) \quad e^{\frac{ik}{2z}(x^2+y^2)} \to 1,$$

then the Fraunhofer approximation holds.

The maximum of (+) is for $\frac{k}{2z}(L_x^2+L_y^2)$

We thus require that

$$\frac{k}{2z}(L_x^2+L_y^2) \ll 1$$

i.e. $$\frac{\pi}{\lambda z}(L_x^2+L_y^2)$$

$$= \frac{\pi\left((20\times10^3)^2 + (40\times10^3)^2\right)}{633 \cdot 2\times10^9} \quad \frac{[nm^2]}{[nm^2]}$$

$= 0.005 \ll 1.$ Appears to be ok.

(e-c)) $$I(\bar{X},\bar{Y},z) = \frac{I_0}{\lambda^2 z^2} \frac{\sin^2((2N+1)k\bar{Y}a/2)}{\sin^2(k\bar{Y}a/2)} L_x^2 L_y^2 \operatorname{sinc}^2(k\bar{X}L_x/2) \cdot \operatorname{sinc}^2(k\bar{Y}L_y/2)$$

$$I(0,\bar{Y},z) = I(0,0,z) \cdot G_{\bar{Y}} \cdot \operatorname{sinc}^2(k\bar{Y}L_y/2)$$

$$G_{\bar{Y}} = \frac{\sin^2((2N+1)k\bar{Y}3L_y)}{\sin^2(k\bar{Y}3L_y)}$$

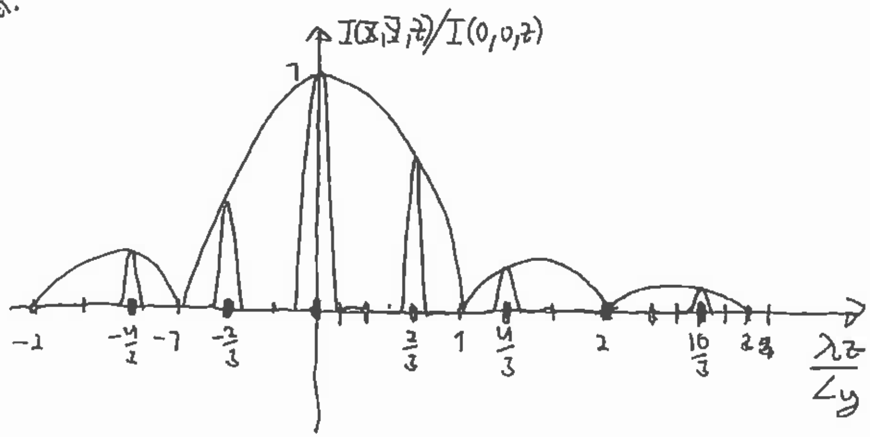
$G_{\bar{Y}} \to (2N+1)^2$ as $k\bar{Y}3L_y = m\pi$, where $m = 0, \pm1, \pm2$

$$k\bar{Y}3L_y \approx \frac{2\pi}{\lambda}\frac{\bar{Y}}{z}3L_y = m\pi$$

$$\Leftrightarrow \bar{Y} = \frac{2}{3}\left(\frac{m}{L_y}\right) z\lambda$$

$\operatorname{sinc}^2(k\bar{Y}L_y/2) = 0$ for $\frac{2\pi}{\lambda}\frac{\bar{Y}}{z}\frac{L_y}{2} = q\pi$

$$\Leftrightarrow \bar{Y} = q\frac{1}{L_y}\lambda z$$


I(x,y,z)/I(0,0,z)
1
-2
-4/3
-1
-2/3
2/3
1
4/3
2
10/3
λz/L_y