NTNU **Institutt for fysikk**

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Exam in TFY4230 STATISTICAL PHYSICS

Wednesday december 1, 2010
15:00–19:00

Allowed help: Alternativ **C**
Standard calculator (according to list prepared by NTNU).
K. Rottman: *Matematisk formelsamling* (all languages).
Barnett & Cronin: Mathematical Formulae
This problem set consists of 2 pages.

Problem 1. Particles in a spherical volume

A system of N classical non-relativistic particles is confined to a spherical (3-dimensional) volume with "soft" walls, described by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{1}{2m}\boldsymbol{p}_i^2 + \varepsilon_0 \left(\frac{\boldsymbol{x}_i^2}{r_0^2}\right)^n, \tag{1}$$

where ε_0 is a positive constant, r_0 is a length characterizing the radius of the sphere, and n is a positive integer.

a) Write down the canonical partition function Z for this system at temperature T.

b) Calculate the internal energy $U = \langle H \rangle$ and heat capacity C for this system.

c) Does your result for C agree with the equipartition theorem when $n = 1$ or $n = \infty$?

d) Calculate the mean particle density, defined as

$$\rho(\boldsymbol{x}) = \left\langle \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i) \right\rangle. \tag{2}$$

Next assume the particles to have charge Q measured in units of the positron charge e, and that the system is exposed to a magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. This implies that we must make the substitution

$$\boldsymbol{p}_i \to \boldsymbol{p}_i + Qe\boldsymbol{A}(\boldsymbol{x}_i) \tag{3}$$

in the Hamiltonian (1).

e) What is the effect of this magnetic field on the classical partition function Z?

The Gamma function:

$$\Gamma(\nu) = \int_0^\infty \frac{\mathrm{d}t}{t}\, t^\nu\, \mathrm{e}^{-t}, \quad \Gamma(1+\nu) = \nu\,\Gamma(\nu), \tag{4}$$

$$\Gamma(1) = 1, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(\nu) = \nu^{-1} + \cdots \text{ when } \nu \to 0. \tag{5}$$

Problem 2. Monte-Carlo simulation of a thermal system

Here you should to prepare for a numerical simulation of the system discussed in the previous problem, for the case of $N = 1$ and $n = 2$. We further simplify the system to be one-dimensional.

a) Write down the classical equations of motion dictated by the Hamiltonian (1).

b) Find suitable units for time and length so that the equations of motion can be written in terms of dimensionless variables.

c) How would you discretize the differential equations for a numerical solution of the problem?

d) To simulate temperature one has to introduce additional fluctuating and a damping forces. Indicate how this should be done.

Hamilton's equations:

$$\dot{\boldsymbol{x}}_\alpha = \frac{\partial H}{\partial \boldsymbol{p}_\alpha}, \qquad \dot{\boldsymbol{p}}_\alpha = -\frac{\partial H}{\partial \boldsymbol{x}_\alpha}. \tag{6}$$

Problem 3. Quantum statistics of thermal radiation

The eigen-energies for the free radiation field can be written

$$E = \sum_{\boldsymbol{k},r} \hbar\omega_{\boldsymbol{k}}\, N(\boldsymbol{k}, r) \tag{7}$$

where $\omega_{\boldsymbol{k}} = c\,|\boldsymbol{k}|$, and where $N(\boldsymbol{k}, r) = 0, 1, \ldots$ is the *occupation number* of the state with wavevector $\boldsymbol{k}$ and polarization r. We have subtracted the zero-point energy. With av volume V and periodic boundary conditions the allowed values for $\boldsymbol{k}$ lie on a lattice,

$$\boldsymbol{k} = \frac{2\pi}{V^{1/3}}\,(n_x, n_y, n_z) \quad \text{with all } n\text{'s integer.} \tag{8}$$

a) Show that the partition function for this system can be written

$$\ln Z = -\sum_{\boldsymbol{k},r} \ln\left(1 - \mathrm{e}^{-\beta\hbar\omega_{\boldsymbol{k}}}\right). \tag{9}$$

b) Explain why the the average occupations numbers can be written as

$$\langle N(\boldsymbol{k}, r)\rangle = -\frac{1}{2}\frac{1}{\beta\hbar}\frac{\partial}{\partial\omega_{\boldsymbol{k}}} \ln Z. \tag{10}$$

c) Find an explicit expression for $\langle N(\boldsymbol{k}, r)\rangle$.

d) To evaluate many physical quantities explicitly in the limit $V \to \infty$ one makes the substitution

$$\sum_{\boldsymbol{k},r} F(\boldsymbol{k}, r) \to V\mathcal{N}\sum_r \int \mathrm{d}^3\boldsymbol{k}\, F(\boldsymbol{k}, r), \tag{11}$$

valid for continuous functions $F(\boldsymbol{k}, r)$.

Explain the origin of this substitution. What is the dimensionless number $\mathcal{N}$?

e) In most of the universe the photons have a temperature $T = 2.725$ K.

How many photons $N = \sum_{\boldsymbol{k},r}\langle N(\boldsymbol{k}, r)\rangle$ are there on average per m^3?

Some physical constants, and an integral:

$$\hbar = 1.054\,571\,628 \times 10^{-34}\ \text{Js}, \quad k_B = 1.380\,6503 \times 10^{-23}\ \text{J/K}, \quad c = 299\,792\,458\text{m/s} \tag{12}$$

$$\int_0^\infty \frac{x^2\mathrm{d}x}{\mathrm{e}^x - 1} = 2\zeta(3) \approx 2.404\ldots \tag{13}$$