NTNU **Institutt for fysikk**

Contact during the exam:
Professor Ingve Simonsen
Telephone: 9 34 17 or 470 76 416

Exam in TFY4275/FY8907 CLASSICAL TRANSPORT THEORY

Jun 08, 2010
09:00–13:00

Allowed help: Alternativ **D**
Authorized calculator and mathematical formula book

This problem set consists of 4=1one page=0.

This exam consists of three problems each containing several sub-problems. Each of the sub-problem will be given approximately equal weight during grading (if noting else is said to indicate otherwise). Problem one may seem long, but there is not much to calculate, so it will probably not take so long to complete. This problem is mostly of conceptual character.

I will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do a round 10am, and the other one, about two hours later.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Good luck to all of you!

Problem 1. "Icelandic" ash problem

On 20 Mar 2010, the volcano *Eyjafjallajökull* (Icelandic for "island-mountain glacier"), situated at the souther part of Island, had a minor eruption. During the morning of 14 Apr 2010, the eruption entered a second phase and created an ash cloud that soon spread over large parts of Europe.

As you probably are fully aware of, these ash clouds caused a very high proportion of flights within, to, and from Europe to be cancelled, creating the highest level of air travel disruption since the Second World War (more then 1000 flights have been canceled so far due to this reason). The cause of the problem is that the jet engines on the planes are not considered to be safe to operate under conditions when the air contains too high levels of ash, since it may damage the engines. Hence, it is important to be able to estimate how the ash will spread.

a) Describe by words the main transport mechanisms for the ash cloud? Which of those mechanisms that you name are expected to dominate?

We will now consider a simplified version of this problem, and we will limit ourselves to two spatial dimensions. The coordinate system is chosen such that the positive x- and y-directions corresponds to east and north, respectively.

b) Write down the diffusion-advection equation in two spatial dimensions where a spatial point is denoted by the distance vector $\mathbf{r}$ [$= (x, y)$] and time by t. Let D denote the diffusion constant, and for the vector field that this equation should contain, use the symbol $\mathbf{v}$ that is assumed to be both time and spatially independent.

For the ash problem, state what the physical meaning is of the dependent variable that you use.

c) Show that the fundamental solution (the propagator)

$$p(\mathbf{r}, t|\mathbf{r}_s, t_s) = \frac{1}{4\pi D(t-t_s)} \exp\left\{ -\frac{[\mathbf{r}-\mathbf{r}_s-\mathbf{v}(t-t_s)]^2}{4D(t-t_s)} \right\}, \qquad t \geq t_s. \tag{1}$$

satisfies the diffusion-advection equation (in two-dimensions). [Hint: Make a change of variables]

What is the meaning of $\mathbf{r}_s$ and t_s in this expression? What are the boundary and initial conditions that this solution satisfies? [Note that you do not have to demonstrate this mathematically].

d) In your own words, explain what is the physical interpretation of the *propagator* (or fundamental solution) given in Eq. (1). What is the expression for $p(\mathbf{r}, t|\mathbf{r}_s, t_s)$ when $t < t_s$?

The initial eruption of the volcano location at $\mathbf{r}_0$ took place at time t_0. Moreover, during the time period from t_0 till t_1 the wind was so weak that it could be neglected (probably not a very realistic situation though). However, at t_1 the wind *suddenly* increased to $\mathbf{v}_1$ and stayed constant from there onward.

During the whole time-period when the volcano was active, the amount of dust per unit time, $\mathcal{A}_0$, emitted by the volcano (located at $\mathbf{r}_0$) was time-independent.

e) Write down an expression for the dust concentration, $c(\mathbf{r}, t)$ (assuming two-dimensional space) where $\mathbf{r}$ denotes an arbitrary spatial point and $t_0 < t \leq t_1$. The expression will contain some integrals that you are not asked to calculate.

f) What is the similar expression for the dust concentration, $c(\mathbf{r}, t)$, when $t > t_1$.

g) Assuming westerly wind and make a drawing of a few contour lines of constant ash concentration for the two cases : (i) $t_0 < t \leq t_1$ and (ii) $t > t_1$. Indicate the location of the volcano and the wind direction. Label the lines a, b and c in order of *decreasing* ash concentration.

Problem 2. Langevin equation

We consider a very small particle of mass, m, that is placed in a thermal bath held at a constant temperature T. The effect of the bath on the particle is described by a (time dependent) *stochastic force*, $\mathbf{S}(t)$. The only properties that will be assumed for the stochastic force are

- $S(t)$ is a *stationary* stochastic process
- $S(t)$ is characterized by a correlation function

$$W(\tau) = \langle \mathbf{S}(t)\mathbf{S}(t+\tau)\rangle_t\,, \tag{2}$$

of general form that has a *finite* correlation time τ_0 [and where $\langle\cdot\rangle_t$ denotes time-average].

When the particle moves about in the bath, a friction force, $\mathbf{F}(t) = m\gamma\mathbf{v}(t)$ will act on the particle where the constant γ is known as the *friction coefficient.*

In this problem, we intend to show by letting you solving several sub-problems, that the friction coefficient, γ, can be expressed in terms of $W(\tau)$ by only assuming $\gamma\tau_0 \ll 1$ in addition to the assumptions made above for $\mathbf{S}(t)$.

a) Describe by words what is the main motivation behind the Langevin equation, and for what systems it is typically used.

b) Write down the Langevin equation for the system in question. Express your equation in terms of the particle velocity, $\mathbf{v}(t)$, the friction coefficient γ and the scaled stochastic force $\boldsymbol{\xi}(t) = \mathbf{S}(t)/m$.

c) Show that the formal solution of this Langevin equation can be written as

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-\gamma t} + e^{-\gamma t}\int_0^t dt'\, e^{\gamma t'}\boldsymbol{\xi}(t'), \tag{3}$$

where $\mathbf{v}(t=0) = \mathbf{v}_0$.

d) Argue why it is physically reasonable to assume that the stochastic force satisfies $\langle\mathbf{S}(t)\rangle = \langle\boldsymbol{\xi}(t)\rangle = 0$.

e) Use this to obtain expressions for the averages (i) $\langle\mathbf{v}(t)\rangle$ and (ii) $\left\langle[\delta\mathbf{v}(t)]^2\right\rangle$ where $\delta\mathbf{v}(t) = \mathbf{v}(t) - \langle\mathbf{v}(t)\rangle$.

f) State the equipartition theorem and argue why it should apply in this particular case in the limit $t \to \infty$. Use it to demonstrate that in this limit ($t \to \infty$) the following relation holds

$$3k_BT\,m = \lim_{t\to\infty} e^{-2\gamma t}\int_0^t dt'\, e^{2\gamma t'}\int_{-t'}^{t-t'} d\tau\, e^{\gamma\tau}W(\tau), \tag{4}$$

where k_B is the Boltzmann constant.

g) Under the assumption that $\gamma\tau_0 \ll 1$, derive an expression for the friction coefficient γ expressed in terms of only $W(\tau)$, T, k_B and m. How can this result be interpreted physically?

Problem 3. Student random walk

Consider a random walk process where the particle after N steps is located at position

$$x_N = \sum_{i=1}^{N} x_i, \tag{5}$$

where the independent and identically distributed (iid's) steps (the x_i's) are drawn from the following probability distribution function (pdf)

$$p(x) = \frac{A}{(1+x^2)^2}, \tag{6}$$

where A is the constant (to be determined).

a) Define what is meant by a characteristic function, $\phi(k)$, of a general pdf $p(x)$.

b) Use this to show that the characteristic function of the distribution in Eq. (6) can be written as

$$\phi(k) = \frac{A\pi}{2}\, e^{-|k|}\,(1+|k|)\,. \tag{7}$$

[Hint: use the residue theorem, and handle $k > 0$ and $k < 0$ separately].

c) Determine the constant A, the mean $\langle x\rangle$ and the variance, σ^2, of the distribution in Eq. (6) [Hint: use $\phi(k)$].

d) Does the random walk process in Eq. (5) correspond to an ordinary-, sub- or super-diffusive process? Give your argument that led you to your conclusion.

e) Write down an expression for the pdf, $p_N(x)$, of the position, x_N, of the walker after N steps expressed in terms of $\phi(k)$. [Note that you are not asked to calculate the integral].

f) Derive an expression for the asymptotic form of $p_N(x)$ in the limit $|x| \to \infty$.