TFY4210 Anvendt kvantemekanikk Vår ~~2004~~ 2006

Løsning øving ~~5~~ 4

Oppgave 1.

a) Å vise hermitisiteten til operatorene $\boldsymbol{A}$, $\boldsymbol{E}$ og $\boldsymbol{B}$ er forholdsvis rett fram. Vi viser det bare for $\boldsymbol{A}$ her, beviset for de to andre er helt tilsvarende. For en hermitesk operator er $\boldsymbol{A}^\dagger = \boldsymbol{A}$, og her er

$$\begin{aligned}\boldsymbol{A}^\dagger &= \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\boldsymbol{k}}}}\left(a_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + a^\dagger_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right)^\dagger \\ &= \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\boldsymbol{k}}}}\left(a^\dagger_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + a_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right) = \boldsymbol{A}.\end{aligned}$$

Som vist i forelesningen er de andre feltene gitt ved

$$\boldsymbol{E} = \mathrm{i}\sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2\varepsilon_0 V}}\left(a_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} - a^\dagger_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

og

$$\boldsymbol{B} = \mathrm{i}\sum_{\boldsymbol{k},\lambda} (\boldsymbol{k}\times\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda})\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\boldsymbol{k}}}}\left(a_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} - a^\dagger_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right),$$

og hermitisiteten av disse ser en lett.

b) For å kunne skrive $\boldsymbol{A}(\boldsymbol{r})$ på den oppgitte måten, lønner det seg først å innse at $\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda} = \hat{\boldsymbol{e}}_{-\boldsymbol{k},\lambda}$ (eller i det minste *kan* man velge basisvektorene slik, og det er ingen grunn til å gjøre det noe annerledes). Dessuten er $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$. Dermed kan en skrive

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r}) &= \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\boldsymbol{k}}}}a_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\boldsymbol{k}}}}a^\dagger_{\boldsymbol{k},\lambda}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \\ &= \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\boldsymbol{k}}}}a_{\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + \sum_{\boldsymbol{k},\lambda} \hat{\boldsymbol{e}}_{-\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{-\boldsymbol{k}}}}a^\dagger_{-\boldsymbol{k},\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \\ &= \sum_{\boldsymbol{k},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\boldsymbol{k}}}}\left(a_{\boldsymbol{k},\lambda} + a^\dagger_{-\boldsymbol{k},\lambda}\right),\end{aligned}$$

altså er

$$\underline{\boldsymbol{A}_{\boldsymbol{k}} = \sum_{\lambda=1}^{2} \hat{\boldsymbol{e}}_{\boldsymbol{k},\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\boldsymbol{k}}}}\left(a_{\boldsymbol{k},\lambda} + a^\dagger_{-\boldsymbol{k},\lambda}\right).}$$

For å vise kommutatorrelasjonen trenger vi kommutatoren

$$\begin{aligned}\left[a_{\boldsymbol{k},\lambda} + a^\dagger_{-\boldsymbol{k},\lambda}, a_{\boldsymbol{k}',\lambda'} + a^\dagger_{-\boldsymbol{k}',\lambda'}\right] &= \left[a_{\boldsymbol{k},\lambda}, a^\dagger_{-\boldsymbol{k}',\lambda'}\right] + \left[a^\dagger_{-\boldsymbol{k},\lambda}, a_{\boldsymbol{k}',\lambda'}\right] \\ &= \delta_{\boldsymbol{k},-\boldsymbol{k}'}\delta_{\lambda,\lambda'} - \delta_{-\boldsymbol{k},\boldsymbol{k}'}\delta_{\lambda,\lambda'} = 0.\end{aligned}$$

Da har vi at

$$
\begin{aligned}
&[\hat{e}_1 \cdot \boldsymbol{A}_{\boldsymbol{k}}, \hat{e}_2 \cdot \boldsymbol{A}_{\boldsymbol{k}'}] \\
&= \left[\sum_{\lambda=1}^{2} (\hat{e}_1 \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda}) \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\boldsymbol{k}}}} \left(a_{\boldsymbol{k},\lambda} + a^{\dagger}_{-\boldsymbol{k},\lambda}\right), \sum_{\lambda'=1}^{2} (\hat{e}_2 \cdot \hat{\mathbf{e}}_{\boldsymbol{k}',\lambda'}) \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\boldsymbol{k}'}}} \left(a_{\boldsymbol{k}',\lambda'} + a^{\dagger}_{-\boldsymbol{k}',\lambda'}\right)\right] \\
&= \sum_{\lambda=1}^{2} \sum_{\lambda'=1}^{2} (\hat{e}_1 \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda})(\hat{e}_2 \cdot \hat{\mathbf{e}}_{\boldsymbol{k}',\lambda'}) \frac{\hbar}{2\varepsilon_0 V \sqrt{\omega_{\boldsymbol{k}} \omega_{\boldsymbol{k}'}}} \left[a_{\boldsymbol{k},\lambda} + a^{\dagger}_{-\boldsymbol{k},\lambda}, a_{\boldsymbol{k}',\lambda'} + a^{\dagger}_{-\boldsymbol{k}',\lambda'}\right] = 0.
\end{aligned}
$$

c) På samme måte som i a) har vi at Fourierkomponentene til de elektriske og magnetiske feltene er

$$
\begin{aligned}
\boldsymbol{E}_{\boldsymbol{k}} &= \mathrm{i} \sum_{\lambda=1}^{2} \hat{\mathbf{e}}_{\boldsymbol{k},\lambda} \sqrt{\frac{\hbar \omega_{\boldsymbol{k}}}{2\varepsilon_0 V}} \left(a_{\boldsymbol{k},\lambda} - a^{\dagger}_{-\boldsymbol{k},\lambda}\right) \\
\boldsymbol{B}_{\boldsymbol{k}} &= \mathrm{i} \sum_{\lambda=1}^{2} (\boldsymbol{k} \times \hat{\mathbf{e}}_{\boldsymbol{k},\lambda}) \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\boldsymbol{k}}}} \left(a_{\boldsymbol{k},\lambda} + a^{\dagger}_{-\boldsymbol{k},\lambda}\right) = \mathrm{i}\boldsymbol{k} \times \boldsymbol{A}_{\boldsymbol{k}}.
\end{aligned}
$$

Dermed blir den første kommutatoren lik

$$
\begin{aligned}
&[\hat{e}_1 \cdot \boldsymbol{E}_{\boldsymbol{k}}, \hat{e}_2 \cdot \boldsymbol{A}_{\boldsymbol{k}'}] \\
&= \left[\mathrm{i} \sum_{\lambda=1}^{2} (\hat{e}_1 \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda}) \sqrt{\frac{\hbar \omega_{\boldsymbol{k}}}{2\varepsilon_0 V}} \left(a_{\boldsymbol{k},\lambda} - a^{\dagger}_{-\boldsymbol{k},\lambda}\right), \sum_{\lambda'=1}^{2} (\hat{e}_2 \cdot \hat{\mathbf{e}}_{\boldsymbol{k}',\lambda'}) \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\boldsymbol{k}'}}} \left(a_{\boldsymbol{k}',\lambda'} + a^{\dagger}_{-\boldsymbol{k}',\lambda'}\right)\right] \\
&= \mathrm{i} \sum_{\lambda=1}^{2} \sum_{\lambda'=1}^{2} (\hat{e}_1 \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda})(\hat{e}_2 \cdot \hat{\mathbf{e}}_{\boldsymbol{k}',\lambda'}) \frac{\hbar}{2\varepsilon_0 V} \sqrt{\frac{\omega_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}'}}} \left[a_{\boldsymbol{k},\lambda} - a^{\dagger}_{-\boldsymbol{k},\lambda}, a_{\boldsymbol{k}',\lambda'} + a^{\dagger}_{-\boldsymbol{k}',\lambda'}\right].
\end{aligned}
$$

Den siste kommutatoren i dette uttrykket blir

$$
\left[a_{\boldsymbol{k},\lambda}, a^{\dagger}_{-\boldsymbol{k}',\lambda'}\right] - \left[a^{\dagger}_{-\boldsymbol{k},\lambda}, a_{\boldsymbol{k}',\lambda'}\right] = 2\delta_{\boldsymbol{k},-\boldsymbol{k}'}\delta_{\lambda,\lambda'},
$$

slik at resultatet blir

$$
\underline{[\hat{e}_1 \cdot \boldsymbol{E}_{\boldsymbol{k}}, \hat{e}_2 \cdot \boldsymbol{A}_{\boldsymbol{k}'}] = \delta_{-\boldsymbol{k}',\boldsymbol{k}} \frac{\mathrm{i}\hbar}{\varepsilon_0 V} \sum_{\lambda=1}^{2} (\hat{e}_1 \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda})(\hat{e}_2 \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda}).}
$$

I den andre kommutatoren kan vi bruke vektoridentiteten $\hat{e}_2 \cdot (\boldsymbol{k} \times \boldsymbol{A}_{\boldsymbol{k}}) = (\hat{e}_2 \times \boldsymbol{k}) \cdot \boldsymbol{A}_{\boldsymbol{k}}$, som gir oss

$$
\begin{aligned}
&[\hat{e}_1 \cdot \boldsymbol{E}_{\boldsymbol{k}}, \hat{e}_2 \cdot \boldsymbol{B}_{\boldsymbol{k}'}] = [\hat{e}_1 \cdot \boldsymbol{E}_{\boldsymbol{k}}, \mathrm{i}\hat{e}_2 \cdot (\boldsymbol{k} \times \boldsymbol{A}_{\boldsymbol{k}})] \\
&= \left[\mathrm{i} \sum_{\lambda=1}^{2} (\hat{e}_1 \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda}) \sqrt{\frac{\hbar \omega_{\boldsymbol{k}}}{2\varepsilon_0 V}} \left(a_{\boldsymbol{k},\lambda} - a^{\dagger}_{-\boldsymbol{k},\lambda}\right), \mathrm{i}(\hat{e}_2 \times \boldsymbol{k}) \cdot \sum_{\lambda'=1}^{2} \hat{\mathbf{e}}_{\boldsymbol{k}',\lambda'} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\boldsymbol{k}'}}} \left(a_{\boldsymbol{k}',\lambda'} + a^{\dagger}_{-\boldsymbol{k}',\lambda'}\right)\right] \\
&= -\delta_{-\boldsymbol{k}',\boldsymbol{k}} \frac{\hbar}{\varepsilon_0 V} \sum_{\lambda=1}^{2} (\hat{e}_1 \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda}) \left[(\hat{e}_2 \times (-\boldsymbol{k})) \cdot \hat{\mathbf{e}}_{-\boldsymbol{k},\lambda}\right]
\end{aligned}
$$

Nå har vi et uttrykk i summen på formen $\sum_{\lambda} (\boldsymbol{v} \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda})(\boldsymbol{w} \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda})$, der $\hat{\mathbf{e}}_{\boldsymbol{k},\lambda}$ for $\lambda \in \{1, 2\}$ er to ortogonale enhetsvektorer. $\boldsymbol{w} = \hat{e}_2 \times \boldsymbol{k}$ er en vektor som ligger i planet utspent av $\hat{\mathbf{e}}_{\boldsymbol{k},1}$ og $\hat{\mathbf{e}}_{\boldsymbol{k},2}$ siden $\boldsymbol{k}$ er ortogonal på dette planet. Derfor er denne summen ganske enkelt lik $\boldsymbol{v} \cdot \boldsymbol{w}$[1].

[1] For å se dette mer eksplisitt; vektorene $\hat{\boldsymbol{k}}$, $\hat{\mathbf{e}}_{\boldsymbol{k},1}$ og $\hat{\mathbf{e}}_{\boldsymbol{k},2}$ er ortogonale basisvektorer i det 3-dimensjonale rommet. Da kan vi skrive $\boldsymbol{v} = \alpha_v \hat{\mathbf{e}}_{\boldsymbol{k},1} + \beta_v \hat{\mathbf{e}}_{\boldsymbol{k},2} + \gamma_v \hat{\boldsymbol{k}}$, og $\boldsymbol{w} = \alpha_w \hat{\mathbf{e}}_{\boldsymbol{k},1} + \beta_w \hat{\mathbf{e}}_{\boldsymbol{k},2}$. Enkel utregning gir da at

$$
\sum_{\lambda} (\boldsymbol{v} \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda})(\boldsymbol{w} \cdot \hat{\mathbf{e}}_{\boldsymbol{k},\lambda}) = \alpha_v \alpha_w + \beta_v \beta_w = \boldsymbol{v} \cdot \boldsymbol{w}.
$$

Dette gir at kommutatoren er

$$\underline{[\hat{e}_1 \cdot \boldsymbol{E}_{\boldsymbol{k}}, \hat{e}_2 \cdot \boldsymbol{B}_{\boldsymbol{k}'}] = \delta_{-\boldsymbol{k}',\boldsymbol{k}} \frac{\hbar}{\varepsilon_0 V} \hat{e}_1 \cdot (\hat{e}_2 \times \boldsymbol{k}) = \delta_{-\boldsymbol{k}',\boldsymbol{k}} \frac{\hbar}{\varepsilon_0 V} (\hat{e}_1 \times \hat{e}_2) \cdot \boldsymbol{k}}$$

Vi har fra ovenfor at $\boldsymbol{B}_{\boldsymbol{k}} = \mathrm{i}\boldsymbol{k} \times \boldsymbol{A}_{\boldsymbol{k}}$. Vi har også vist at alle Fourierkomponentene $\boldsymbol{A}_{\boldsymbol{k}}$ kommuterer innbyrdes, derav følger det umiddelbart at

$$\underline{[\hat{e}_1 \cdot \boldsymbol{B}_{\boldsymbol{k}}, \hat{e}_2 \cdot \boldsymbol{A}_{\boldsymbol{k}'}] = 0}.$$

Siden Fourierkomponentene av det elektriske og det magnetiske feltet ikke kommuterer, vil det ikke være mulig å lage et elektromagnetisk felt med spesifiserte verdier av begge feltene overalt.