Løsningsforslag, Øving 5, TFY 4210 (1)

Oppgave 1 Vi må som vanlig beregne

$$w_{i\to f} = \frac{2\pi}{\hbar}\,\frac{V\, m\, p_f}{h^3}\, d\Omega\, |M|^2$$

der

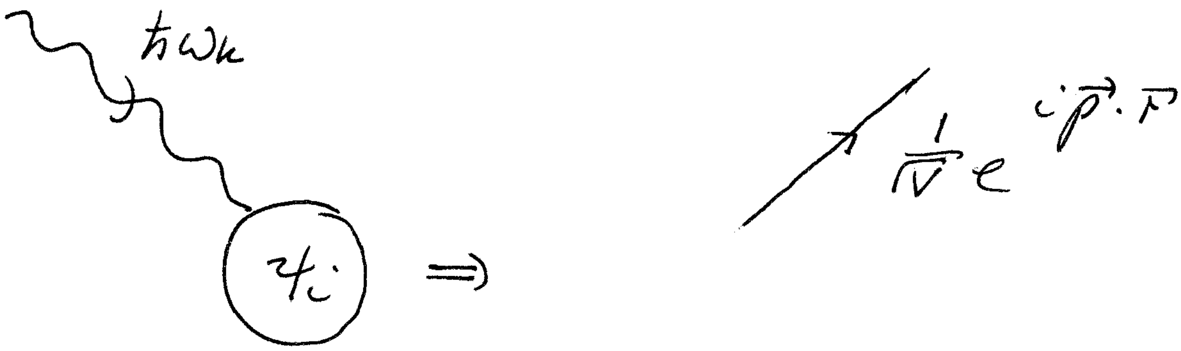
$$M = \langle F | H_I | I \rangle$$

$$H_I = \frac{e}{m}\,\vec{A}\cdot\vec{p}$$

$|I\rangle$ er en initiell elektron-foton tilstand, $|F\rangle$ er en elektron-foton slutt-tilstand.

$$|I\rangle = |\psi_i\rangle\,|\,\text{Inn-fotoner}\,\rangle$$

$$|F\rangle = |\psi_f\rangle\,|\,\text{Ut-fotoner}\,\rangle$$



Foton med bølgetall $\vec{k}$ absorberes

$$\left.\begin{aligned} |i\rangle &= |\;\cdots, n_{k}, \cdots\rangle \\ |f\rangle &= |\cdots, n_{k} \cdots\rangle \end{aligned}\right\} \langle f | a_{k\lambda} | i \rangle : \text{Absorbsjon}$$

$$\vec A = \frac{1}{\sqrt{\varepsilon_0 V}} \sum_{\vec k,\lambda} \sqrt{\frac{\hbar}{2\omega_k}} \Big[a_{k\lambda}(t)\, e^{i\vec k\cdot\vec r} + a^+_{k\lambda}(t)\, e^{-i\vec k\cdot\vec r} \Big] \vec e_{k\lambda}$$

$$a_{k\lambda}(t) = a_{k\lambda}\, e^{i\omega_{k\lambda} t}$$

Energi-bevarelse: $\hbar\omega_{k\lambda} = |E_0| + \frac{p_f^2}{2m}$

der $|E_0|$ er bindingsenergien, og

$\frac{p_f^2}{2m}$ er kinetisk energi i slutt-tilstand.

Matrise-elementet M blir dermed:

$$M = \frac{e}{m}\frac{1}{\sqrt{\varepsilon_0 V}} \sqrt{\frac{\hbar}{2\omega_k}} \left(\cancel{\sqrt{n_k}}\right)$$

$$\vec e_{k\lambda} \cdot \int d\vec r\, \frac{1}{\sqrt V}\, e^{-i\vec p\cdot\vec r/\hbar}\, e^{i\vec k\cdot\vec r}\, \frac{\hbar}{i}\nabla \psi_i$$

$$= \frac{e}{m}\frac{1}{\sqrt{\varepsilon_0 V}} \sqrt{\frac{\hbar}{2\omega_k}} \sqrt{n_k+1}$$

$$(\vec e_{k\lambda}\cdot \vec p) \int d\vec r\, e^{-i\vec q\cdot\vec r}\, \psi_i(\vec r)$$

der $\hbar\vec q = \vec p - \hbar\vec k$ :: impulsoverføringen i støtet mellom fotonet og elektronet.

$$\psi_i = \frac{1}{\sqrt{32\pi a^3}}\, r\cos\theta\, e^{-\frac{r}{2a}}$$

$$\int d\vec{r}\, e^{-i\vec{q}\cdot\vec{r}}\, \psi_i(\vec{r})$$

$$= \frac{1}{\sqrt{32\pi a^3}}\, 2\pi \int_0^\pi d\theta \sin\theta \int_0^\infty dr\, r^2\, r\cos\theta\, e^{-\frac{r}{2a}} \cdot e^{-iqr\cos\theta}$$

Gjør først θ-integrasjonen

$x = -\cos\theta$

$dx = \sin\theta\, d\theta$

$$-\int_{-1}^{1} dx\, x\, e^{iqrx}$$

$$= -\frac{\partial}{\partial(iqr)} \underbrace{\int_{-1}^{1} dx\, e^{iqrx}}_{\frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)}$$

$$= \frac{2\sin(qr)}{qr}$$

$$= 2i\, \frac{\partial}{\partial(qr)} \left[\frac{\sin(qr)}{qr}\right]$$

$$= 2i\left[\frac{\cos(qr)}{qr} - \frac{\sin qr}{(qr)^2}\right]$$

$$\frac{4\pi i}{\sqrt{32\pi a^3}} \int_0^\infty dr\, r^3 \left[\frac{\cos(qr)}{qr} - \frac{\sin(qr)}{(qr)^2}\right] e^{-\frac{r}{2a}}$$

$$= \frac{4\pi i}{\sqrt{32\pi a^3}} \left(\frac{I_1}{q} - \frac{I_2}{q^2}\right)$$

$$I_1 = \int_0^\infty dr\, r^2 \cos(qr)\, e^{-\frac{r}{2a}}$$

Vi ser: $I_1 = \frac{\partial I_2}{\partial q}$

$$I_2 = \int_0^\infty dr\, r \sin(qr)\, e^{-\frac{r}{2a}}$$

$$\cancel{I_1 = \frac{\partial I_2}{\partial q} = 2a \frac{\partial I_2}{\partial(2aq)} = (2a)^2 \frac{\left[1-(2aq)^2\right]}{\left[1+(2aq)^2\right]^2}}$$

$$\cancel{I_2 = 2a \frac{2aq}{1+(2aq)^2}}$$

Se korrigerte integraler, side 7 og 8

Dermed har vi:

$$M = \frac{e}{m}\frac{1}{\sqrt{\varepsilon_0 V}}\sqrt{\frac{\hbar}{2\omega_k}}\underbrace{\left[\sqrt{n_k}\ \text{[redacted]}\right]}_{NB!!}\left(\vec{e}_{k\lambda}\cdot\vec{q}\right)$$

$$\cdot \frac{4\pi i}{\sqrt{32\pi a^3}} \left(\frac{I_1}{q} - \frac{I_2}{q^2}\right)$$

Vi ser at når $n_k = 0$, får vi ingen fotoelektrisk effekt. Vi får ikke elektroner slynget ut av atomet spontant.

Oppgave 2

Koherent tilstand: $a|\alpha\rangle = \alpha|\alpha\rangle$

$\langle\alpha|H|\alpha\rangle$

$$= \hbar\omega \langle\alpha|(a^+a + \tfrac{1}{2})|\alpha\rangle$$

$$= \frac{\hbar\omega}{2} + \hbar\omega\langle\alpha|a^+a|\alpha\rangle$$

$$= \underline{\underline{\hbar\omega\left(|\alpha|^2 + \tfrac{1}{2}\right)}}$$

$\langle\alpha|H^2|\alpha\rangle$

$$= (\hbar\omega)^2 \langle\alpha|(a^+a + \tfrac{1}{2})(a^+a + \tfrac{1}{2})|\alpha\rangle$$

$$= (\hbar\omega)^2\langle\alpha|(a^+a\,a^+a + a^+a + \tfrac{1}{4})|\alpha\rangle$$

$$= (\hbar\omega)^2\langle\alpha||\alpha|^2\,a\,a^+ + |\alpha|^2 + \tfrac{1}{4}|\alpha\rangle$$

$$= (\hbar\omega)^2\left(2|\alpha|^2 + \tfrac{1}{4} + |\alpha|^4\right)$$

$$= (\hbar\omega)^2\left(2|\alpha|^2 + \tfrac{1}{4} + |\alpha|^4\right)$$

$$(\langle\alpha|H|\alpha\rangle)^2 = (\hbar\omega)^2\left[|\alpha|^4 + \tfrac{1}{4} + |\alpha|^2\right]$$

$$\langle\alpha|H^2|\alpha\rangle - (\langle\alpha|H|\alpha\rangle)^2$$

$$= (\hbar\omega)^2\left[2|\alpha|^2 + \tfrac{1}{4} + |\alpha|^4 - |\alpha|^4 - \tfrac{1}{4} - |\alpha|^2\right]$$

$$= \underline{\underline{(\hbar\omega)^2\,|\alpha|^2}}$$

Korrigerte integraler I_1 og I_2 i oppg. 1 (7)

$$I_2 = -\frac{\partial}{\partial q} \underbrace{\int_0^\infty dr \cos(qr)\, e^{-\frac{r}{2a}}}_{I_0}$$

$$I_0 = \mathrm{Re} \int_0^\infty e^{-r\left(\frac{1}{2a} - iq\right)}$$

$$= \mathrm{Re}\left(\frac{1}{\frac{1}{2a} - iq}\right) = \mathrm{Re}\left(\frac{2a}{1 - i2qa}\right)$$

$$= 2a \quad \frac{2qa}{1+(2qa)^2}$$

$$I_2 = -(2a)^2 \frac{\partial}{\partial x}\left(\frac{x}{1+x^2}\right)\Big|_{x=2qa}$$

$$= -(2a)^2 \quad \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}$$

$$= -(2a)^2 \quad \frac{(1-x^2)}{(1+x^2)^2} \quad ; \; x = 2qa$$

$$I_1 = -(2a)^3 \frac{\partial}{\partial x}\left(\frac{1-x^2}{(1+x^2)^2}\right)\Big|_{x=2qa}$$

$$= -(2a)^3 \left\{\frac{-2x}{(1+x^2)^2} - \frac{\cancel{2}(1-x^2) \cdot 2x}{(1+x^2)^3}\right\}$$

$$= -(2a)^3 \, \frac{(-2x)}{(1+x^2)^2} \left(1 + \frac{1-x^2}{1+x^2}\right)$$

$$= (2a)^3 \, \frac{2\cancel{x} \cdot 2}{(1+x^2)^3} \; ; \qquad x = 2a$$