Oppg. 1 Øving 7, TFY4210, Vår 2006

$$\frac{\partial u}{\partial t} = -\frac{i}{\hbar} \hat{V}(t)\, u(t,t_0)$$

$$u(t,t_0) = 1 + \sum_{n=1}^{\infty} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \; \hat{V}(t_1)\hat{V}(t_2)\cdots\hat{V}(t_n)$$

Vi skal nå få dette over på en form slik at alle øvre grenser er like. Vi gjør dette ved å gå „motsatt" vei, og starter med 2. ordens ledd:

$$\frac{1}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \tilde{T}[\hat{V}(t_1)\hat{V}(t_2)]$$

Innfører tidsordnings-symbolet $\tilde{T}$.

$$\tilde{T}[\hat{V}(t_1)\hat{V}(t_2)] = \begin{cases} V(t_1)V(t_2); & t_1 > t_2 \\ V(t_2)V(t_1); & t_2 > t_1 \end{cases}$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \hat{V}(t_1)\hat{V}(t_2)$$

$$+ \frac{1}{2}\int_{t'}^{t} dt_2 \int_{t'}^{t_2} dt_1\, \hat{V}(t_2)\hat{V}(t_1)$$

I 2. ledd: $\left.\begin{matrix} t_2 \to t_1 \\ t_1 \to t_2 \end{matrix}\right\}$ Omdøping av variable

Da blir 2. ledd identisk med 1. ledd, og vi får:

$$\frac{1}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; \tilde{T}\left[\hat{V}(t_1)\hat{V}(t_2)\right]$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; \hat{V}(t_1)\hat{V}(t_2)$$

På samme måte:

$$\frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \; \tilde{T}\left[\hat{V}(t_1)\cdots\hat{V}(t_n)\right]$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \; \hat{V}(t_1)\cdots\hat{V}(t_n)$$

Dermed finner vi:

$$\mathcal{U}(t,t_0) = 1 + \sum_{n=1}^{\infty} \frac{(-\frac{i}{\hbar})^n}{n!} \cdot \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \; \tilde{T}\left[\hat{V}(t_1)\cdots\hat{V}(t_n)\right]$$

$$= \tilde{T}\left\{ e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt' \, \hat{V}(t')} \right\} \quad \text{Formel!}$$

Legg merke til at dette er vesentlig mer komplisert enn $e^{-\frac{i}{\hbar}V(t-t')}$!!

Oppgave 2, Øv. 7, TFY4210, Vår 2006

$$(*) \quad \left(\frac{dS}{dx}\right)^2 - 2m\left(E - V(x)\right) - i\hbar \frac{d^2S}{dx^2} = 0$$

$$S = S_0 + \hbar S_1 + \frac{\hbar^2}{2} S_2 + \cdots$$

$$\frac{dS}{dx} = \frac{dS_0}{dx} + \hbar \frac{dS_1}{dx} + \frac{\hbar^2}{2}\frac{dS_2}{dx} + \cdots$$

$$\frac{d^2S}{dx^2} = \frac{d^2S_0}{dx^2} + \hbar \frac{d^2S_1}{dx^2} + \frac{\hbar^2}{2}\frac{d^2S_2}{dx^2} + \cdots$$

$$\left(\frac{dS}{dx}\right)^2 = \cancel{\frac{dS_0}{dx}} \left(S_0' + \hbar S_1' + \frac{\hbar^2}{2} S_2' + \cdots\right)^2$$

$$= (S_0')^2 + \hbar^2 (S_1')^2 + 2\hbar S_0' S_1' + \hbar^2 S_0' S_2' + \cdots$$

(til orden $\hbar^2$)

Setter alt dette inn i $(*)$, og regner orden for orden i $\hbar$

$$(S_0')^2 + \hbar^2 (S_1')^2 + 2\hbar S_0' S_1' + \hbar^2 S_0' S_2' + \cdots$$
$$- 2m\left(E - V(x)\right)$$
$$- i\hbar \left(S_0'' + \hbar S_1'' + \frac{\hbar^2}{2} S_2'' + \cdots\right) = 0$$

$$\mathcal{O}(\hbar^0): \quad (S_0')^2 - 2m\left(E - V(x)\right) = 0$$

$$\mathcal{O}(\hbar^1): \quad S_0' S_1' - \frac{i}{2} S_0'' = 0$$

$$\mathcal{O}(\hbar^2): \quad (S_1')^2 + S_0' S_2' - i S_1'' = 0$$

$$S_0' = \pm p(x) \quad ; \quad p(x) = \sqrt{2m\left(E - V(x)\right)}$$

$$S_0 = \pm \int^{x} dx' \, p(x')$$

$$S_1' = \frac{i}{2} \frac{S_0''}{S_0'}$$

$$= \frac{i}{2} \frac{d}{dx} \left(\ln p(x) \right)$$

$$S_1 = \frac{i}{2} \ln\left[p(x) \right]$$

$$S_2' = i \frac{S_1''}{S_0'} \div \frac{(S_1')^2}{S_0'}$$

$$= -\frac{1}{2} \frac{S_0''}{(S_0')^2} + \frac{1}{(S_0')^3} \left(S_0'' \right)^2$$

$$S_0' = \pm p(x)$$

$$S_0'' = \pm \frac{-2m \frac{dV}{dx}}{2 p(x)} = - m \frac{1}{p(x)} \frac{dV}{dx}$$

Første ledd i S_2':

$$-\frac{1}{2} \frac{S_0''}{(S_0')^2} = \frac{1}{2} \frac{d}{dx} \left(\frac{1}{S_0'} \right)$$

$$S_2(x) = \int^{x} dx' \left[\frac{m}{2} \frac{1}{p^3(x')} \frac{dV}{dx'} + \frac{m^2}{4} \frac{1}{p^5(x')} \left(\frac{dV}{dx'} \right)^2 \right]$$

Vi ser at S_2 er liten når $\left|\frac{dV}{dx}\right|$ er liten, og $p(x)$ ikke er nær null, dvs. vi er borte fra klassiske vendepunkter.