Øving 10 - Løsningsforslag

Problem 9.37

(a) Equation 9.91 $\Rightarrow \tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$; $\mathbf{k}_T\cdot\mathbf{r} = k_T(\sin\theta_T\,\hat{\mathbf{x}} + \cos\theta_T\,\hat{\mathbf{z}})\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) =$ $k_T(x\sin\theta_T + z\cos\theta_T) = xk_T\sin\theta_T + izk_T\sqrt{\sin^2\theta_T - 1} = kx + i\kappa z$, where

$$k \equiv k_T\sin\theta_T = \left(\frac{\omega n_2}{c}\right)\frac{n_1}{n_2}\sin\theta_I = \frac{\omega n_1}{c}\sin\theta_I,$$

$$\kappa \equiv k_T\sqrt{\sin^2\theta_T - 1} = \frac{\omega n_2}{c}\sqrt{(n_1/n_2)^2\sin^2\theta_I - 1} = \frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2}.\quad \text{So}$$

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}.\quad \text{qed}$$

$\beta \equiv \dfrac{\mu_1 n_2}{\mu_2 n_1}$ $\qquad$ $\alpha \equiv \dfrac{\cos\theta_T}{\cos\theta_I}$

(b) $R = \left|\dfrac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}}\right|^2 = \left|\dfrac{\alpha-\beta}{\alpha+\beta}\right|^2$. Here β is real (Eq. 9.106) and α is purely imaginary (Eq. 9.108); write $\alpha = ia$, with a real: $R = \left(\dfrac{ia-\beta}{ia+\beta}\right)\left(\dfrac{-ia-\beta}{-ia+\beta}\right) = \dfrac{a^2+\beta^2}{a^2+\beta^2} = \boxed{1.}$

(c) From Prob. 9.16, $E_{0_R} = \left|\dfrac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0_I}$, so $R = \left|\dfrac{1-\alpha\beta}{1+\alpha\beta}\right|^2 = \left|\dfrac{1-ia\beta}{1+ia\beta}\right|^2 = \dfrac{(1-ia\beta)(1+ia\beta)}{(1+ia\beta)(1-ia\beta)} = \boxed{1.}$

(d) From the solution to Prob. 9.16, the transmitted wave is

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}},\quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_T\,\hat{\mathbf{x}} + \sin\theta_T\,\hat{\mathbf{z}}).$$

Disse må forutsettes gitt

Using the results in (a): $\mathbf{k}_T\cdot\mathbf{r} = kx + i\kappa z - \omega t$, $\sin\theta_T = \dfrac{ck}{\omega n_2}$, $\cos\theta_T = i\dfrac{c\kappa}{\omega n_2}$:

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\,\hat{\mathbf{y}},\quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\left(-i\frac{c\kappa}{\omega n_2}\,\hat{\mathbf{x}} + \frac{ck}{\omega n_2}\,\hat{\mathbf{z}}\right).$$

We may as well choose the phase constant so that $\tilde{E}_{0_T}$ is *real*. Then

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{y}};\\
\mathbf{B}(\mathbf{r},t) &= \frac{1}{v_2}E_0 e^{-\kappa z}\frac{c}{\omega n_2}\mathrm{Re}\left\{[\cos(kx-\omega t) + i\sin(kx-\omega t)]\,[-i\kappa\,\hat{\mathbf{x}} + k\,\hat{\mathbf{z}}]\right\}\\
&= \frac{1}{\omega}E_0 e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\,\hat{\mathbf{x}} + k\cos(kx-\omega t)\,\hat{\mathbf{z}}\right].\quad \text{qed}
\end{aligned}$$

(I used $v_2 = c/n_2$ to simpify $\mathbf{B}$.)

(e) (i) $\nabla\cdot\mathbf{E} = \dfrac{\partial}{\partial y}\left[E_0 e^{-\kappa z}\cos(kx-\omega t)\right] = 0.$ ✓

(ii)
$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \frac{\partial}{\partial x}\left[\frac{E_0}{\omega}e^{-\kappa z}\kappa\sin(kx-\omega t)\right] + \frac{\partial}{\partial z}\left[\frac{E_0}{\omega}e^{-\kappa z}k\cos(kx-\omega t)\right]\\
&= \frac{E_0}{\omega}\left[e^{-\kappa z}\kappa k\cos(kx-\omega t) - \kappa e^{-\kappa z}k\cos(kx-\omega t)\right] = 0.\ ✓
\end{aligned}$$

(iii)
$$\begin{aligned}
\nabla\times\mathbf{E} &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & E_y & 0\end{vmatrix} = -\frac{\partial E_y}{\partial z}\,\hat{\mathbf{x}} + \frac{\partial E_y}{\partial x}\,\hat{\mathbf{z}}\\
&= \kappa E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{x}} - E_0 e^{-\kappa z}k\sin(kx-\omega t)\,\hat{\mathbf{z}}.\\
-\frac{\partial\mathbf{B}}{\partial t} &= -\frac{E_0}{\omega}e^{-\kappa z}\left[-\kappa\omega\cos(kx-\omega t)\,\hat{\mathbf{x}} + k\omega\sin(kx-\omega t)\,\hat{\mathbf{z}}\right]\\
&= \kappa E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{x}} - kE_0 e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{z}} = \nabla\times\mathbf{E}.\ ✓
\end{aligned}$$

(iv)
$$\begin{aligned}
\nabla\times\mathbf{B} &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ B_x & 0 & B_z\end{vmatrix} = \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\hat{\mathbf{y}}\\
&= \left[-\frac{E_0}{\omega}\kappa^2 e^{-\kappa z}\sin(kx-\omega t) + \frac{E_0}{\omega}e^{-\kappa z}k^2\sin(kx-\omega t)\right]\hat{\mathbf{y}} = (k^2-\kappa^2)\frac{E_0}{\omega}e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{y}}.
\end{aligned}$$

Eq. 9.202 $\Rightarrow k^2 - \kappa^2 = \left(\dfrac{\omega}{c}\right)^2\left[n_1^2\sin^2\theta_I - (n_1\sin\theta_I)^2 + (n_2)^2\right] = \left(\dfrac{n_2\omega}{c}\right)^2 = \omega^2\epsilon_2\mu_2.$

$$\begin{aligned}
&= \epsilon_2\mu_2\omega E_0 e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{y}}.\\
\mu_2\epsilon_2\frac{\partial\mathbf{E}}{\partial t} &= \mu_2\epsilon_2 E_0 e^{-\kappa z}\omega\sin(kx-\omega t)\,\hat{\mathbf{y}} = \nabla\times\mathbf{B}\ ✓.
\end{aligned}$$

(f)
$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_2}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_2}\frac{E_0^2}{\omega}e^{-2\kappa z}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & \cos(kx-\omega t) & 0\\ \kappa\sin(kx-\omega t) & 0 & k\cos(kx-\omega t)\end{vmatrix}\\
&= \boxed{\frac{E_0^2}{\mu_2\omega}e^{-2\kappa z}\left[k\cos^2(kx-\omega t)\,\hat{\mathbf{x}} - \kappa\sin(kx-\omega t)\cos(kx-\omega t)\,\hat{\mathbf{z}}\right].}
\end{aligned}$$

Averaging over a complete cycle, using $\langle\cos^2\rangle = 1/2$ and $\langle\sin\cos\rangle = 0$, $\langle\mathbf{S}\rangle = \dfrac{E_0^2 k}{2\mu_2\omega}e^{-2\kappa z}\,\hat{\mathbf{x}}$. On average, then, no energy is transmitted in the z direction, only in the x direction (parallel to the interface). qed