Øving 11 - Løsningsforslag

Problem 9.24

Equation 9.170 $\Rightarrow n = 1 + \frac{Nq^2}{2m\epsilon_0}\frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}$. Let the denominator $\equiv D$. Then

$\frac{dn}{d\omega} = \frac{Nq^2}{2m\epsilon_0}\left\{\frac{-2\omega}{D} - \frac{(\omega_0^2-\omega^2)}{D^2}\left[2(\omega_0^2-\omega^2)(-2\omega)+\gamma^2 2\omega\right]\right\} = 0 \Rightarrow 2\omega D = (\omega_0^2-\omega^2)\left[2(\omega_0^2-\omega^2)-\gamma^2\right]2\omega;$

$(\omega_0^2-\omega^2)^2+\gamma^2\omega^2 = 2(\omega_0^2-\omega^2)^2 - \gamma^2(\omega_0^2-\omega^2)$, or $(\omega_0^2-\omega^2)^2 = \gamma^2(\omega^2+\omega_0^2-\omega^2) = \gamma^2\omega_0^2 \Rightarrow (\omega_0^2-\omega^2) = \pm\omega_0\gamma;$

$\omega^2 = \omega_0^2 \mp \omega_0\gamma$, $\omega = \omega_0\sqrt{1\mp\gamma/\omega_0} \cong \omega_0(1\mp\gamma/2\omega_0) = \omega_0 \mp \gamma/2$. So $\omega_2 = \omega_0+\gamma/2$, $\omega_1 = \omega_0-\gamma/2$, and the width of the anomalous region is $\boxed{\Delta\omega = \omega_2-\omega_1 = \gamma.}$

From Eq. 9.171, $\alpha = \frac{Nq^2\omega^2}{m\epsilon_0 c}\frac{\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$, so at the maximum ($\omega=\omega_0$), $\alpha_{\max} = \frac{Nq^2}{m\epsilon_0 c\gamma}$.

At ω_1 and ω_2, $\omega^2 = \omega_0^2\mp\omega_0\gamma$, so $\alpha = \frac{Nq^2\omega^2}{m\epsilon_0 c}\frac{\gamma}{\gamma^2\omega_0^2+\gamma^2\omega^2} = \alpha_{\max}\left(\frac{\omega^2}{\omega^2+\omega_0^2}\right)$. But

$\frac{\omega^2}{\omega^2+\omega_0^2} = \frac{\omega_0^2\mp\omega_0\gamma}{2\omega_0^2\mp\omega_0\gamma} = \frac{1}{2}\frac{(1\mp\gamma/\omega_0)}{(1\mp\gamma/2\omega_0)} \cong \frac{1}{2}\left(1\mp\frac{\gamma}{\omega_0}\right)\left(1\pm\frac{\gamma}{2\omega_0}\right) \cong \frac{1}{2}\left(1\mp\frac{\gamma}{2\omega_0}\right) \cong \frac{1}{2}.$

So $\alpha \cong \frac{1}{2}\alpha_{\max}$ at ω_1 and ω_2. qed

Problem 9.25

$k = \frac{\omega}{c}\left[1+\frac{Nq^2}{2m\epsilon_0}\sum\frac{f_j}{(\omega_j^2-\omega^2)}\right]$. $v_g = \frac{d\omega}{dk} = \frac{1}{(dk/d\omega)}$.

$\frac{dk}{d\omega} = \frac{1}{c}\left[1+\frac{Nq^2}{2m\epsilon_0}\sum\frac{f_j}{(\omega_j^2-\omega^2)} + \omega\sum f_j\frac{-(-2\omega)}{(\omega_j^2-\omega^2)^2}\right] = \frac{1}{c}\left[1+\frac{Nq^2}{2m\epsilon_0}\sum f_j\frac{(\omega_j^2+\omega^2)}{(\omega_j^2-\omega^2)^2}\right].$

$\boxed{v_g = c\left[1+\frac{Nq^2}{2m\epsilon_0}\sum f_j\frac{(\omega_j^2+\omega^2)}{(\omega_j^2-\omega^2)^2}\right]^{-1}.}$ Since the second term in square brackets is *positive*, it follows that

$\boxed{v_g < c,}$ whereas $v = \frac{\omega}{k} = c\left[1+\frac{Nq^2}{2m\epsilon_0}\sum\frac{f_j}{(\omega_j^2-\omega^2)}\right]^{-1}$ is greater than c or less than c, depending on ω.

Problem 9.28

Here $a = 2.28\,\text{cm}$ and $b = 1.01\,\text{cm}$, so $\nu_{10} = \frac{1}{2\pi}\omega_{10} = \frac{c}{2a} = 0.66\times10^{10}\,\text{Hz}$; $\nu_{20} = 2\frac{c}{2a} = 1.32\times10^{10}\,\text{Hz}$; $\nu_{30} = 3\frac{c}{2a} = 1.97\times10^{10}\,\text{Hz}$; $\nu_{01} = \frac{c}{2b} = 1.49\times10^{10}\,\text{Hz}$; $\nu_{02} = 2\frac{c}{2b} = 2.97\times10^{10}\,\text{Hz}$; $\nu_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = 1.62\times10^{10}\,\text{Hz}$. Evidently just four modes occur: $\boxed{10, 20, 01, \text{ and } 11.}$

To get only *one* mode you must drive the waveguide at a frequency between ν_{10} and ν_{20}: $\boxed{0.66\times10^{10} < \nu < 1.32\times10^{10}\,\text{Hz}.}$ $\lambda = \frac{c}{\nu}$, so $\lambda_{10} = 2a$; $\lambda_{20} = a$. $\boxed{2.28\,\text{cm} < \lambda < 4.56\,\text{cm}.}$